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6 SEM TDC MTMH (CBCS) C 14

2024

(May)

MATHEMATICS

(Core)

Paper : C–14

(Ring Theory and Linear Algebra—II)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer any *three* from the following : 5×3=15

(a) Define polynomial ring and prove that if D is an integral domain, then $D[x]$ is also an integral domain.

(b) Let F be a field. Then prove that $F[x]$ is principal ideal domain.

(c) State division algorithm for $F[x]$ and find the quotient and remainder upon dividing $f(x) = 3x^4 + x^3 + 2x^2 + 1$ by $g(x) = x^2 + 4x + 2$ where $f(x)$ and $g(x)$ belong to $Z_5[x]$.

(d) State Eisenstein's criterion on irreducibility. And prove that, in an integral domain, every prime is irreducible.

2. Answer any *three* from the following : 5×3=15

(a) Prove that if F is a field, then $F[x]$ is Euclidean domain.

(b) Prove that every ideal of Euclidean domain is principal ideal.

(c) Prove that every principal ideal domain is unique factorization domain.

(d) Show that the ring

$$Z[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a,\ b \in Z\}$$

is an integral domain but not a unique factorization domain.

3. Answer any *three* from the following : 6×3=18

(a) Let V be an n-dimensional vector space over the field F and let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for V. Let $B' = \{f_1, f_2, \ldots, f_n\}$ be the dual basis of B. Then prove that—

(i) for each linear functional f on V, $f = \sum_{i=1}^{n} f(\alpha_i) f_i$;

(ii) for each vector α in V, $\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$.

(b) Find the dual basis of the basis set

$$B = \{(1, -2, 3), (1, -1, 1), (2, -4, 7)\}$$

of $V_3(R)$.

(c) Let W_1 and W_2 be subspaces of a finite dimensional vector space V. Then prove that—

(i) $(W_1 + W_2)^\circ = W_1^\circ \cap W_2^\circ$;

(ii) $(W_1 \cap W_2)^\circ = W_1^\circ + W_2^\circ$.

(d) Find the minimal polynomial of the real matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Also, show that minimal polynomial of a matrix or of a linear operator is unique.

4. *(a)* Show that the space generated by (1, 1, 1) and (1, 2, 1) is an invariant subspace of R^3 under T, where

$$T(x, y, z) = (x+y-z,\ x+y,\ x+y-z)$$ 3

(b) Prove that the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable over the field C. 4

Or

Find all complex eigenvalues and eigenspaces of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

5. *(a)* If α and β are vectors in an inner product space $V(F)$ and $a, b \in F$, then prove that—

(i) $\|a\alpha + b\beta\|^2 = |a|^2\|\alpha\|^2 + a\bar{b}(\alpha, \beta) + \bar{a}b(\beta, \alpha) + |b|^2\|\beta\|^2$;

(ii) $\text{Re}(\alpha, \beta) = \frac{1}{4}\|\alpha+\beta\|^2 - \frac{1}{4}\|\alpha-\beta\|^2$. 5

(b) If α and β are vectors in a real inner product space and if $\|\alpha\| = \|\beta\|$, then prove that $\alpha - \beta$ and $\alpha + \beta$ are orthogonal and interpret the result geometrically. 3+2=5

(c) Given the basis (2, 0, 1), (3, −1, 5) and (0, 4, 2) for $V_3(R)$. Construct from it by the Gram-Schmidt process an orthonormal basis relative to the standard inner product space. 5

Or

Let V be a finite dimensional inner product space and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an orthonormal basis for V. Show that for any vectors $\alpha, \beta \in V$,

$$(\alpha, \beta) = \sum_{k=1}^{n} (\alpha, \alpha_k)\overline{(\beta, \alpha_k)}$$

6. *(a)* If T is skew, does it follow that so is T^2? What about T^3? 1+1=2

(b) Answer any *two* from the following : 4×2=8

(i) Let V be the vector space $V_2(C)$ with the standard inner product. Let T be the linear operator defined by

$$T(1, 0) = (1, -2),\quad T(0, 1) = (i, -1)$$

If $\alpha = (a, b)$, then find $T^*\alpha$.

(ii) Prove that a linear transformation E is an orthogonal projection if and only if $E = E^2 = E^*$.

(iii) Prove that a necessary and sufficient condition that a self-adjoint linear transformation T on an inner product space V be $\hat{0}$ is that $(T\alpha, \alpha) = 0,\ \forall\ \alpha \in V$.

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