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4 SEM TDC MTMH (CBCS) C 9

2024

(May/June)

MATHEMATICS

(Core)

Paper : C–9

(Riemann Integration and Series of Functions)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* State Riemann condition of integrability. 1

   (b) If f is bounded on $[a, b]$ and M, m are supremum and infimum of f on $[a, b]$, then prove that

$$m(b-a) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le M(b-a)$$ 4

2. (a) Define partition and tagged partition of a closed interval $[a, b]$. 1+1=2

(b) Let f be a bounded function defined on $[a, b]$. If Q is a refinement of a partition P, then prove that

$$U(f, P) \geq U(f, Q)$$ 3

Or

Prove that a bounded function $f \in R[a, b]$, if for $\varepsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon$$

(c) Answer any *four* of the following questions : 5×4=20

(i) Give an example with explanation that a function which is Riemann integrable but neither monotonic nor continuous.

(ii) Prove that a continuous function is integrable.

(iii) Let $f, g \in R[a, b]$. Prove that $f + g$ is integrable and that

$$\int_a^b (f + g)\, dx = \int_a^b f dx + \int_a^b g dx$$

(iv) Let $f : [a, b] \to \mathscr{R}$ be differentiable and f' is integrable on $[a, b]$. Then prove that

$$\int_a^b f'(x) dx = f(b) - f(a)$$

(v) Show that if a function f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\, dx = f(c)(b - a)$$

3. (a) Show that the improper integral

$$\int_0^\infty e^{-x} dx$$

exists. 2

(b) Show that

$$\int_0^1 x^{n-1}e^{-x}\,dx$$

is convergent, if $n > 0$. 3

(c) Prove that

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$ 5

Or

Prove that

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

4. (a) What is the difference between pointwise convergence and uniform convergence? 1

(b) Give an example to prove that if a sequence of functions $\{f_n\}$ converges uniformly to a function f, then it converges pointwise to f also. But the converse is not true. 2

(c) Show that the function defined by

$$f_n(x) = \begin{cases} 0, & -\infty < x \le 0 \\ nx, & 0 \le x \le \frac{1}{n} \\ 1, & x \ge \frac{1}{n} \end{cases}$$

converges pointwise to $f(x) = 0$ for $x \le 0$ and $f(x) = 1$ for $x > 0$. 4

(d) Show that if $f_n : X \to \mathscr{R}$ be a sequence of uniformly convergent functions, then the sequence $\{f_n\}$ is uniformly Cauchy on X. 4

(e) Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$, and $f_n \to f$ uniformly on $[a, b]$. Prove that f is continuous and hence integrable on $[a, b]$. Hence show that

$$\int_a^b f(x)\,dx = \lim \int_a^b f_n(x)\,dx$$ 4

(f) Find the pointwise limit of the sequence of real-valued function

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad x \in [0, 1]$$ 5

(g) State and prove Weierstrass M-test for the series of functions. 5

5. (a) Define limit interior and limit superior. 1+1=2

(b) Let R is the radius of convergence of the power series

$$\sum a_n x^n$$

Prove that the series is absolutely convergent if $|x| < R$ and divergent if $|x| > R$. 4

(c) Find the radius of convergence of the power series

$$\frac{x}{2} + \frac{1.3}{2.5}x^2 + \frac{1.3.5}{2.5.8}x^3 + \cdots$$ 4

(d) If a power series

$$\sum a_n x^n$$

converges to a particular value $x_0 \neq 0$, then show that it converges absolutely for every x for which $|x| \leq |x_0|$. 5

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