Total No. of Printed Pages—16

4 SEM TDC GEMT (CBCS) 4.1/4.2/4.3

2024

(May/June)

MATHEMATICS

(Generic Elective)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

All symbols have their usual meanings

Paper : GE—4.1

(Algebra)

UNIT—1

1. *(a)* State True or False : 1

   Addition of natural numbers in binary composition is not associative.

   (b) Find the elements of U(20). 1

   (c) Show that the subset $\{1, -1, i, -i\}$ of the complex numbers is an Abelian group under complex multiplication. 5

(d) Prove that in a group G, $(ab)^{-1} = b^{-1}a^{-1}$, for all $a, b \in G$. 3

(e) Prove that a group in which every element is its own inverse is Abelian. 4

2. *(a)* Define quaternion group. 1

(b) Describe the symmetries of an isosceles triangle. 3

(c) Prove that the set of matrices

$$A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

where α is a real number, forms a group under matrix multiplication. 5

Or

Prove that every permutation of a finite set can be written as a product of disjoint cycles.

(d) Prove that the order of a cyclic group is equal to the order of its generator. 5

Or

Show that {1, 2, 3} under multiplication modulo 4 is not a group but that {1, 2, 3, 4} under multiplication modulo 5 is a group.

UNIT—2

3. *(a)* State True or False : 1

Order of a cyclic group is not equal to the order of its generator.

(b) Write all the left cosets of H in G if $G = S_3$ and $H = \{1, (13)\}$. 3

(c) Prove that a non-empty subset H of a group G is a subgroup of G iff $a, b \in H \Rightarrow ab^{-1} \in H$. 5

(d) Show that the centre of a group G is a subgroup of G. 5

Or

Prove that a subgroup of a cyclic group is cyclic.

4. *(a)* Prove that every subgroup of an Abelian group is normal. 2

(b) State and prove Lagrange's theorem. 4

(c) Prove that every quotient group of a cyclic group is cyclic. 4

(d) Prove that if p is a prime number, then any group G of order $2p$ has a normal subgroup of order p. 4

Or

Let G be a group and let H be a normal subgroup of G. Prove that the set $G/H = \{ah : a \in G\}$ is a group under the operation $(aH)(bH) = abH$.

UNIT—3

5. (a) Define zero divisor. 1

(b) State True or False : 1

A commutative ring R is called an integral domain if R has zero divisor.

(c) Prove that the set $R = \{0, 1, 2, 3, 4, 5\}$ is a commutative ring with respect to addition and multiplication modulo 6 as the two-ring composition. 5

(d) Prove that a ring R is without zero divisor if and only if the cancellation laws hold in R. 5

Or

Prove that every field is an integral domain.

6. (a) Define division ring. 1

(b) Show that a ring of order p^2 (p is a prime) may not be commutative. 3

(c) Show that the set of numbers of the form $a + b\sqrt{2}$, with a and b as rational numbers, is a field. 4

(d) Show that, a non-empty subset S of a ring R is a subring of R iff $x, y \in S \Rightarrow xy, x - y \in S$. 4

Or

Show that a commutative ring with unity is a field if it has no proper ideal.

Paper : GE—4.2

(Application of Algebra)

1. তলৰ যি কোনো দুটা প্ৰশ্নৰ উত্তৰ দিয়া : 6×2=12

Answer any *two* of the following questions :

(a) ধৰাহওক, V, mটা মৌল থকা এটা সংহতি আৰু $D = V^{(k)}$ হৈছে kটা মৌল থকা Vৰ সকলো উপ-সংহতিৰ সংহতি, $1 < k < m$. তেনেহ'লে দেখুওৱা যে D হৈছে (m, b, r, k, λ) প্ৰাচলৰ সৈতে Vৰ এটা BIBD, য'ত

Let V be a set of m elements, and let $D = V^{(k)}$ be the set of all subsets of V having k elements, $1 < k < m$. Then show that D is a BIBD on V with parameters (m, b, r, k, λ), where

$$b = \binom{m}{k};\ r = \binom{m-1}{k-1};\ \lambda = \binom{m-2}{k-2}$$

(b) যদি (v, b, r, k, λ) প্ৰাচলৰ সৈতে এটা BIBDৰ অস্তিত্ব থাকে তেন্তে তলৰ সম্বন্ধবোৰ প্ৰমাণ কৰা :

If there exists a BIBD with parameters (v, b, r, k, λ), then prove the following :

(i) $vr = bk$

(ii) $r(k-1) = \lambda(v-1)$

(iii) $b > r > \lambda$

(c) ধৰাহওক, A এটা BIBDৰ ইন্সিডেন্স মেট্ৰিক্স। তেন্তে প্ৰমান কৰা যে AA^T এটা অক্ষীয়মান মেট্ৰিক্স।

Let A be the incidence matrix of a BIBD. Then prove that AA^T is a non-singular matrix.

2. *(a)* যদি $S = \{1, 2, 4\}$, (7, 3, 1) প্ৰাচলৰ সৈতে যোগাত্মক গ্ৰুপ $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$ৰ অন্তৰ সংহতি হয়, তেন্তে BIBDবোৰ নিৰ্ণয় কৰা। 4

Find the BIBDs determined by the difference set $S = \{1, 2, 4\}$ in the additive group $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$ with parameter (7, 3, 1).

অথবা / *Or*

(b) দ্বিঘাত ৰেচিডিউ মডুল' 11ৰ সংহিতটো নিৰ্ণয় কৰা আৰু ইয়াৰ দ্বাৰা সমমিত BIBD এটা গঠন কৰা।

Find the set of quadratic residues modulo 11, and construct the symmetric BIBD determined by it.

3. তলৰ যিকোনো দুটা প্ৰশ্নৰ উত্তৰ দিয়া : 6×2=12

Answer any *two* of the following questions :

(a) ধৰাহওক, C ন্যূনতম দৈৰ্ঘ্য dৰ সৈতে এটা ক'ড আৰু $t = \left[\frac{d-1}{2}\right]$, সৰ্বোচ্চ অখণ্ড সংখ্যা $\leq \frac{d-1}{2}$. তেন্তে প্ৰমাণ কৰা যে—

Let C be a code with minimum distance d. Let $t = \left[\frac{d-1}{2}\right]$ denote the greatest integer $\leq \frac{d-1}{2}$. Then prove that—

(i) Cয়ে ট্ৰেন্সমিট কৰা ক'ডৱৰ্ডৰ ভুলবোৰ $d-1$লৈকে ধৰা পেলাব পাৰে;

C can detect up to $d-1$ errors in any transmitted codeword;

(ii) Cয়ে ট্ৰেন্সমিট কৰা ক'ডৱৰ্ডৰ ভুলবোৰ tলৈকে ধৰা পেলাব পাৰে।

C can detect up to t errors in any transmitted codeword.

(b) এটা প্ৰমাণ কৰা যে, Ham(r, q) হৈছে ন্যূনতম দৈৰ্ঘ্য 3ৰ সৈতে এটা নিখুঁত ক'ড।

Prove that Ham(r, q) is a perfect code with minimum distance 3.

(c) চাইক্লিক ক'ডৰ সংজ্ঞা দিয়া। ধৰাহওক, G হৈছে ৰৈখিক $[n, k]$-ক'ড Cৰ এটা জেনেৰেটৰ মেট্ৰিক্স। তেন্তে প্ৰমাণ কৰা যে C এটা চাইক্লিক ক'ড হ'ব যদি কেৱল যদিহে $\sigma(G_i) \in C$ হয় প্ৰত্যেক শাৰী Gৰ G_iৰ বাবে।

Define cyclic code. Let G be a generator matrix of a linear $[n, k]$-code C. Then prove that C is a cyclic code if and only if $\sigma(G_i) \in C$ for each row G_i of G.

4. $\mathbb{F}_{11}$ত $[10, 8]$-ক'ড Cৰ ন্যূনতম দৈৰ্ঘ্য সমতা পৰীক্ষা কৰা মেট্ৰিক্স Hৰ সৈতে নিৰ্ণয় কৰা, য'ত 4

Find the minimum distance of the $[10, 8]$-code C over $\mathbb{F}_{11}$ with parity-check matrix H, where

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{bmatrix}$$

5. *(a)* ধৰাহওক, $\alpha, \beta \in S_x$ এটা বিচ্ছিন্ন বিন্যাস। তেন্তে প্ৰমাণ কৰা যে $\alpha\beta = \beta\alpha$. 4

Let $\alpha, \beta \in S_x$ be disjoint permutations. Then prove that $\alpha\beta = \beta\alpha$.

(b) এটা সংহতিৰ ওপৰত এটা গ্ৰুপৰ ক্ৰিয়াৰ বিষয়ে চমু টোকা লিখা। 4

Write a short note on the action of a group on a set.

(c) পলিয়াৰ উপপাদ্যটো উল্লেখ আৰু প্ৰমাণ কৰা। 8

State and prove Polya's theorem.

অথবা / Or

চাৰিটা শীৰ্ষ বিন্দুৰ অসমৰূপী গ্ৰাফৰ বাবে জেনেৰেটিং ফলন $f_4(x)$ নিৰ্ণয় কৰা।

Find the generating function $f_4(x)$ for the non-isomorphic graphs on four vertices.

6. (a) $Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$ পজিটিভ ডেফিনিট হয় নে নহয়, পৰীক্ষা কৰা। 4

Check whether

$Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$

is positive definite or not.

(b) $x^Tx = 1$ চৰ্ত সাপেক্ষে

$$Q(x) = 9x_1^2 + 4x_2^2 + 3x_3^2$$ৰ

সৰ্ব্বোচ্চ আৰু সৰ্বনিম্ন মান নিৰ্ণয় কৰা। 6

Find the maximum and minimum values of $Q(x) = 9x_1^2 + 4x_2^2 + 3x_3^2$ subject to the constraint $x^Tx = 1$.

(c) ধৰাহওক, A হৈছে ৰেংক rৰ সৈতে $m \times n$ৰ এটা মেট্ৰিক্স। তেন্তে দেখুওৱা যে তাত $m \times n$ মেট্ৰিক্স, $\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ থাকে, য'ত Dত ডায়োগনেল মৌলবোৰ প্ৰথম rটা Aৰ একক মান $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$; আৰু $m \times m$ অৰ্থোগোনেল মেট্ৰিক্স U আৰু $n \times n$ অৰ্থোগনেল মেট্ৰিক্স Vৰ বাবে $A = U\Sigma V^T$. 6

Let A be an $m \times n$ matrix with rank r. Then show that there exists an $m \times n$ matrix $\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ for which the diagonal entries in D are the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and there

exists an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that $A = U\Sigma V^T$.

7. (a) $Ax = b$ৰ লিষ্ট-স্কোৱৰৰ সমাধান উলিওৱা

য'ত $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ আৰু $b = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$ 8

Find a least squares solution of $Ax = b$ for

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

(b) তলৰ মেট্ৰিক্সটো ৰ'-ৰিদিউসদ এশ্বিলন ফৰ্মলৈ নিবলৈ ৰ'-ৰিদাকশ্যন এলগ'ৰিথম ব্যৱহাৰ কৰা : 8

Use row-reduction algorithm to reduce the following matrix into row-reduced echelon form :

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Paper : GE—4.3

(Combinatorial Mathematics)

1. (a) Find ${}^6P_2$. 1

(b) State the principle of inclusion. 1

(c) A student has 3 pens and 2 pencils. In how many ways can he take a pen and pencil? 2

(d) How many 2-digit numbers can be formed using 3, 4, 5, 6, 7? 2

(e) For a set of six true or false questions, find the number of ways of answering all questions. 2

(f) Find the number of distinguishable words that can be formed from the letters of MADAM. 2

(g) Show that ${}^nC_r = {}^nC_{n-r}$. 2

2. (a) Write the principle of pigeonhole. 2

(b) How many integers between 1 and 300 are

(i) divisible by 3 or 5;

(ii) divisible by 3, but not by 5 or 6?

2+2=4

(c) Let A and B be subsets of a finite universal set U. Then show that

$$|A+B| = |A| + |B| - |A \cap B|$$ 4

Or

Show that if any five numbers from 1 to 8 are chosen, then two of them will add to 9.

3. (a) Define a generating function. 2

(b) Find a generating function to count the number of integral solutions to $e_1 + e_2 + e_3 = 10$ if for each i, $e_i \geq 0$. 2

(c) Answer any *two* questions of the following : 4×2=8

(i) Show that the exponential generating function for the sequence $(1, 1\cdot3, 1\cdot3\cdot5, 1\cdot3\cdot5\cdot7, \cdots)$ is $(1-2x)^{-\frac{3}{2}}$.

(ii) Find the binomial generating function for the sequence

$$a = 1, 2, 3, \cdots, r, \cdots$$

(iii) Find the sequences corresponding to the ordinary generating functions $(3+x)^3$, $(3x^3 + e^{2x})$ and $2x^2(1-x)^{-1}$.

4. (a) Write about a recurrence relation. 1

(b) Solve the recurrence relation $a_n = a_{n-1} + 3$ with $a_1 = 2$. 2

(c) Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with initial conditions $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$. 3

(d) Find the explicit formula for the Fibonacci number. 4

Or

Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

5. (a) Write the number of partitions of 5. 1

(b) Find the ordinary generating function of the sequence $< C(r+n-1, n-1) >_{r \geq 0}$. 2

(c) Find the coefficient of x^7 in $(1 + x + x^2 + \cdots)^{15}$. 3

(d) Find the number of positive integral solutions to the equation $x + y + z = 10$. 6

Or

Find the values of the extended binomial coefficients

$$\binom{-2}{3} \text{ and } \binom{\frac{1}{2}}{3}$$

6. (a) Determine the cycle index of the alternative group $A(n)$. 2

(b) Show that there are precisely 17824 distinguishable (under rotations) vertex colourings of the regular dodecahedron using 1 or 2 colours. 4

(c) Find the number of distinguishable necklaces consisting of 7 stones of which 2 stones are red, 3 stones are blue, 2 stones are green when both rotational and reflectional symmetries are considered. 6

7. (a) What do you mean by a symmetric BIBD? 1

(b) Illustrate the procedure for the group of subsets of $X = \{a, b\}$ under the symmetric difference. 5

Or

Find the number of (rotationally) distinct ways of painting the faces of a cube using 6 colours, so that each face is of different colours.

(c) Prove that, in case of a symmetric BIBD, any two blocks have λ treatment in common. 6

Or

Find how many different necklaces having 10 beads can be formed using 2 different kinds of beads, if *(i)* both flips and rotations and *(ii)* rotations only considered.

★★★