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2 SEM TDC MTMH (CBCS) C 3

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(May)

MATHEMATICS

(Core)

Paper : C–3

(Real Analysis)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* State True or False : 1

The supremum and infimum of a set may or may not belong to the set.

(b) Prove that the supremum of a non-empty set S of real numbers whenever it exists is unique. 3

(c) If $S = \left\{ \frac{4n+3}{n} : n \in \mathbb{N} \right\}$, find $\inf S$ and $\sup S$ if they exist. 2

(d) Let S and T be non-empty subsets of $\mathbb{R}$ with the property $s \leq t$, $\forall\, s \in S$ and $t \in T$. Show that S is bounded above and T is bounded below. 4

Or

Prove that the countable union of countable sets is countable.

(e) Prove that a point p is a limit point of a set S if and only if every neighbourhood of p contains infinitely many points of p. 4

(f) State and prove the Archimedean property of real numbers. 4

Or

Prove that for any real number x, there exists an unique integer m, such that $m \leq x < m+1$.

(g) State the order completeness property of real numbers. 2

(h) Show that if $a < b$, then $a < \frac{1}{2}(a+b) < b$. 3

(i) If x and y are any real numbers with $x < y$, then prove that there exists an irrational number z such that $x < z < y$. 3

(j) If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$, is a nested sequence of closed bounded intervals, then prove that there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$. 4

Or

Prove that there exists a real number x, such that $x^2 = 2$.

2. (a) Write an example of a constant sequence. 1

(b) Show that the sequence $\left\langle \frac{n}{n+1} \right\rangle$ is bounded, $\forall\, n \in \mathbb{N}$. 3

(c) Prove that every convergent sequence has a unique limit. 3

(d) Write the 3-tail of the sequence

$\langle 2, 4, 6, 8, 10, \ldots, 2n, \ldots \rangle$ 1

(e) Let $\langle x_n \rangle$ be a sequence of real numbers and let $x \in \mathbb{R}$. If $\langle a_n \rangle$ is a sequence of positive real numbers with $\underset{n\to\infty}{\text{Lt}} \langle a_n \rangle = 0$ and if for some constant $c > 0$ and some $m \in \mathbb{N}$, $|x_n - x| \le c a_n$ for all $n \ge m$, then prove that

$$\underset{n\to\infty}{\text{Lt}} \langle x_n \rangle = x$$ 4

Or

Prove that

$$\underset{n\to\infty}{\text{Lt}} (n^{1/n}) = 1$$

(f) If $X = \langle x_n \rangle$ is a convergent sequence of real numbers and if $x_n \ge 0$, for all $n \in \mathbb{N}$, then prove that

$$x = \underset{n\to\infty}{\text{Lt}} \langle x_n \rangle \ge 0$$ 4

Or

If $X = \langle x_n \rangle$ and $Y = \langle y_n \rangle$ are convergent sequences of real numbers, and $x_n \le y_n$ for all $n \in \mathbb{N}$, then prove that

$$\underset{n\to\infty}{\text{Lt}} \langle x_n \rangle \le \underset{n\to\infty}{\text{Lt}} \langle y_n \rangle$$

(g) Let $X = \langle x_n \rangle$ be a sequence such that $\underset{n\to\infty}{\text{Lt}} \langle x_n \rangle = x$. Prove that

$$\underset{n\to\infty}{\text{Lt}} \langle |x_n| \rangle = |x|$$ 4

(h) State the properties for a sequence $\langle x_n \rangle$ of real numbers to be divergent. 2

(i) State monotone subsequence theorem. 2

(j) Using Cauchy's criterion of convergence, establish the convergence or divergence of any *two* of the following sequences : 3×2=6

(i) Sequence $\langle x_n \rangle$, where $x_n = \frac{(-1)^n}{n}$

(ii) Sequence $\langle x_n \rangle$, where

$$x_n = 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1}$$

(iii) Sequence $\langle x_n \rangle$, where

$$x_n = 1 + \frac{1}{4} + \frac{1}{7} + \dots + \frac{1}{3n-2}$$

3. (a) Fill in the blanks : 2

An infinite series in which all the terms are of the same sign is _____ if each term is _____ than some finite quantity however small.

(b) State Cauchy's general principle of convergence for series and show that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge. 1+3=4

(c) Prove that a positive term series converges if and only if the sequence of its partial sums is bounded above. 3

Or

Investigate the behaviour of the infinite series whose nth term is $\frac{1}{\lfloor n}$.

(d) State d'Alembert's ratio test.

Is d'Alembert's ratio test stronger than Cauchy's root test? 2+1=3

(e) Show that for any fixed value of x, the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is convergent. 4

(f) Show that the series

$$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

converges absolutely for all values of x. 4

Or

Test the absolute convergence of the series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

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