5 SEM TDC DSE MTH (CBCS)
2.1/2.2/2.3/2.4 (H)

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(November)

MATHEMATICS

(Discipline Specific Elective)

(For Honours)

Paper : DSE–2.1/2.2/2.3/2.4

The figures in the margin indicate full marks
for the questions

Paper : DSE–2.1

(MATHEMATICAL MODELLING)

Full Marks : 60

Pass Marks : 24

Time : 3 hours

1. *(a)* Find the value of $\Gamma\left(\dfrac{5}{2}\right)$. 1

(b) Is the point $x = 0$ an ordinary point of the equation

$$y'' + x^2 y' + x^{1/2} y = 0 ?$$ 1

(Turn Over)

(c) Find the value of $L\{\cosh kx\}$. 　2

(d) Find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s-a)}$$ 　3

2. (a) Investigate the nature of the point $t=0$ for the differential equation

$$t^4 y'' + (1 - \cos t)y + (t^2 \sin t)y' = 0$$ 　2

(b) Find a power series solution of the differential equation $(x-3)y' + 2y = 0$. Determine the radius of convergence of the resulting series. 　4+1=5

3. (a) Use Laplace transforms to solve the initial value problem

$$x'' - x' - 6x = 0;\ x(0) = 2,\ x'(0) = -1$$ 　5

Or

$$x'' + 8x' + 15x = 0;\ x(0) = 2,\ x'(0) = -3$$

(b) Find the Frobenius series solutions of

$$xy'' + 2y' + xy = 0$$ 　6

4. (a) Define 'goal programs'. 　1

(b) Use the linear congruence method to generate 10 random numbers using $a = 5$, $b = 1$ and $c = 8$. 　2

(c) Why is sensitivity analysis important in linear programming? 　3

5. (a) Define feasible solution. 　1

(b) Solve the following model algebraically : 　4

Maximize $25x + 30y$

subject to

$$20x + 30y \le 690$$
$$5x + 4y \le 120$$
$$x,\ y \ge 0$$

(c) Explain middle-square method and use it to generate random numbers taking $x_0 = 2041$. Does this method have any drawbacks? Illustrate. 　2+3+1=6

6. Answer any *three* of the following questions : 　6×3=18

(a) Using Monte Carlo simulation, write an algorithm to calculate the volume of the sphere $x^2 + y^2 + z^2 \le 1$ that lies in the first octant, $x > 0$, $y > 0$, $z > 0$.

(4)

(b) Use the golden-section search method with a tolerance of $t = 0.2$ to minimize

$$f(x) = x^2 + 2x, \; -3 \le x \le 6$$

(c) Use the curve-fitting criterion to minimize the sum of the absolute deviations for the model $y = ax^3$ and data set

| x | : | 7 | 14 | 21 | 28 | 35 | 42 |
| y | : | 8 | 41 | 133 | 250 | 280 | 297 |

(d) Using simplex method, solve the following problem :

Maximize $3x + y$

subject to

$$2x + y \le 6$$
$$x + 3y \le 9$$
$$x, y \ge 0$$

(e) Fit the model $y = cx$ to the following data using Chebyshev's criterion to minimize the largest deviation :

| x | : | 1 | 2 | 3 |
| y | : | 2 | 5 | 8 |

(5)

Paper : DSE–2.2

(MECHANICS)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

UNIT—I

1. (a) Define couple moment. A force

$$\vec{F} = (10\hat{i} + 6\hat{j} + 3\hat{k})$$

acts at position $(3, 0, 2)$. At point $(0, 2, -3)$, an equal but opposite force $-\vec{F}$ acts. Calculate the couple moment.
1+2=3

(b) Show that any number of coplanar couples acting on a body is equivalent to a single couple whose moment is equal to the algebraic sum of moments of the couples.   6

Or

Derive the general equations of equilibrium.

(c) An electric light fixture weighing 15 N hangs from a point C by two strings AC and BC. The string AC is inclined at 60° to the horizontal and BC at 45° to the horizontal. Draw the free-body diagram and determine forces in the strings AC and BC.   5

(6)

(d) A parallel system of forces is such that a 20-N force acts at position $x = 10$ m, $y = 3$ m; a 30-N force acts at position $x = 5$ m, $y = -3$ m; a 50-N force acts at position $x = -2$ m, $y = 5$ m.

(i) If all forces act along negative z-direction, give the simplest resultant force and its line of action.

(ii) If the 50-N force acts along positive z-direction and the others act along negative direction, find the resultant. 4+2=6

UNIT—II

2. (a) What do you mean by coefficient of friction? Write down the dimension of coefficient of friction. 2+2=4

(b) The horizontal position of the 500 kg rectangular block of concrete is adjusted by the 5° wedge under the action of the force P. If the coefficient of friction for both wedge surfaces is 0·30 and the coefficient of friction between horizontal surface and block is 0·60, then determine the least force P required to move the block. 6

(7)

Or

A strongbox of mass 75 kg rests on a floor. The static coefficient of friction for the contact surface is 0·20. Calculate the largest force P and highest position h for applying this force that will not allow the strongbox to either slip on the floor or to tip.

(c) Find the centroid of the area under the half-sine wave. 5

(d) Show that

$$I_{xx} = \frac{bh^3}{12}, \ I_{yy} = \frac{b^3 h}{12} \text{ and } I_{xy} = \frac{b^2 h^2}{24}$$

for the right-angled triangle of base b and height h. 5

(e) State and prove the theorem of Pappus-Guldinus. 5

UNIT—III

3. (a) Write down the statement of law of conservation of mechanical energy. 2

(b) Define constant force field. 1

(c) Given the following conservative force field

$$\vec{F} = (10z + y)\hat{i} + (15yz + x)\hat{j} + \left(10x + \frac{15y^2}{2}\right)\hat{k}$$

Find the force potential. Also, calculate the work done by $\vec{F}$ on a particle going from

$$\vec{r_1} = 10\hat{i} + 2\hat{j} + 3\hat{k} \text{ to } \vec{r_2} = -2\hat{i} + 4\hat{j} - 3\hat{k}$$

$$4+3=7$$

Or

Show that the moment of the resultant force on a particle about a point, fixed in an inertial reference, is equal to the time rate of change of moment of the linear momentum of the particle relative to the inertial reference frame. 7

4. (a) State and prove Chasles' theorem. 7

(b) A cylinder of radius R rotates about its own axis with an angular speed ω. If the total mass is M, show that the kinetic energy is

$$\frac{1}{4}MR^2\omega^2$$

6

(Continued)

(c) Establish the relation between acceleration vectors of a particle for two systems of references moving arbitrarily relative to each other. 6

(d) Derive the moment of momentum equation for a single particle. 6

Paper : DSE–2.3

(NUMBER THEORY)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

1. (a) Exhibit a complete residue system modulo 7 consisting of multiples of 3. 2

(b) If $\tau(n) = 2$, then what conclusion can be drawn for the number n? 1

(c) Write the values of $\mu(49)$ and $\left[-\frac{1}{3}\right]$. 2

(d) How many primitive roots of 9 are there? 1

(Turn Over)

(e) Find the value of the Legendre symbol

$$\left(\frac{2}{17}\right)$$

1

(f) What do you understand by primitive root of an integer n? 1

2. (a) Find the solutions in positive integers for any *one* of the following : 5

(i) $18x + 5y = 48$

(ii) $15x + 7y = 111$

(b) Prove that $a^{21} \equiv a(\text{mod } 15)$ for all a. 3

3. Answer any *two* of the following : $5 \times 2 = 10$

(a) Arrange the integers $2, 3, 4, \cdots, 21$ in pairs a and b that satisfy

$$ab \equiv 1(\text{mod } 23)$$

(b) Solve the following simultaneous congruences :

$x \equiv 5(\text{mod } 7)$

$x \equiv 7(\text{mod } 11)$

$x \equiv 3(\text{mod } 13)$

(c) State and prove Wilson's theorem. Is the converse of Wilson's theorem true?

4. (a) What is meant by number theoretic function? When is it said to be a multiplicative function? 1+1=2

(b) Find the value of $\tau(900)$. 3

(c) Prove that the Mobius μ function is a multiplicative function. 3

(d) Prove that for a prime p

$$\phi(p^k) = p^k \left(1 - \frac{1}{p}\right)$$

Hence find $\phi(32)$. 4

5. Answer any *three* of the following : $5 \times 3 = 15$

(a) For each positive integer $n \geq 1$, prove that

$$n = \sum_{d|n} \phi(d)$$

(b) State and prove Mobius inversion formula.

(c) For $n > 1$, prove that the sum of the positive integers less than n and relatively prime to n is

$$\frac{1}{2}n\phi(n)$$

(d) If $\gcd(m, n) = 1$, then prove that the set of positive divisors of mn consists of all products $d_1 d_2$ where $d_1 \mid m$, $d_2 \mid n$ with $\gcd(d_1, d_2) = 1$.

6. (a) Write the conditions for which an integer n fails to have primitive root.   2

(b) Define the words 'plaintext' and 'ciphertext'.   2

(c) Determine whether 2 is quadratic residue or non-residue of 13.   3

(d) Find a Pythagorean triple of the form 16, y, z.   2

(e) Let p be an odd prime and $(a, p) = 1$. Prove that

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}$$
   3

7. Answer any *three* of the following :   5×3=15

(a) Solve :

$$x^2 + 7x + 10 \equiv 0 \pmod{11}$$

(b) If order of a modulo $p = 3$, where p is prime, then show that order of $(a+1)$ modulo p is 6.

(c) If p is an odd prime, then there are precisely $\frac{(p-1)}{2}$ quadratic residues and $\frac{(p-1)}{2}$ quadratic non-residues of p.

Prove it.

(d) Solve :

$$x^2 \equiv 7 \pmod{3^3}$$

(e) Prove that there are infinitely many primes of the form $8k - 1$.

Paper : DSE–2.4

(BIOMATHEMATICS)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

UNIT—I

1. Answer any *two* of the following questions :

$7\frac{1}{2}\times2=15$

(a) A population is originally 100 individuals, but because of the combined effects of births and deaths, it triples each hour.

 (i) Make a table of population size for $t=0$ to 5, where t is measured in hours.

 (ii) Give two equations modeling the population growth by first expressing P_{t+1} in terms P_t and then expressing ΔP in terms of P_t.

 (iii) What can you say about the birth and death rates for this population?

(b) In the early stages of the development of a frog embryo, cell division occurs at a fairly regular rate. Suppose you observe that all cells divide, and hence

(Continued)

the number of cells double, roughly every half-hour.

 (i) Write down an equation modeling this situation. You should specify how much real-world time is required by an increment of 1 in t and what the initial number of cells is.

 (ii) Produce a table and graph of the number of cells as a function of t.

(c) Obtain a simple prey-predator model explaining in detail the assumptions taken. Also find the equilibrium positions.

UNIT—II

2. Answer any *two* of the following questions :

$7\frac{1}{2}\times2=15$

(a) Consider the SI epidemic model. If the contact rate is 0·001 and the number of susceptibles is 2000 initially, determine—

 (i) the number of susceptibles left after 3 weeks;

 (ii) the density of susceptible when the rate of appearance of new cases is a maximum;

(Turn Over)

(iii) the time (in week) at which the rate of appearance of new cases is a maximum;

(iv) the maximum rate of appearance of new cases.

(b) In an SIS model, if the infection is spread only by a constant number of carriers, then show that

$$I(t) = \left(I_0 - \frac{\alpha CN}{\alpha C + \beta} \right) e^{[-(\alpha C + \beta)t]} + \frac{\alpha CN}{\alpha C + \beta}$$

where I and C are the number of infectives and carriers; N is total population; α and β are contact rate and susceptible rate respectively; I_0 is the infectives at $t = 0$.

(c) Let x and y respectively denote the proportion of susceptibles and carriers in a population. Suppose the carriers are indentified and removed from the population at a rate β, so that

$$\frac{dy}{dt} = \beta y$$

Suppose also that the disease spreads at a rate proportional to the product of x and y, thus

$$\frac{dx}{dt} = -\alpha xy$$

(Continued)

(i) Determine the proportions of carriers at any time t, where $y(0) = y_0$.

(ii) Use *(i)* above to find the susceptibles at time t, where $x(0) = x_0$.

(iii) Find the proportion of population that escapes the epidemic.

UNIT—III

3. Answer any *two* of the following questions : $7\frac{1}{2} \times 2 = 15$

(a) Consider the competition models for two species with populations N_1 and N_2

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12} \frac{N_2}{K_1} \right)$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - b_{21} \frac{N_1}{K_2} \right)$$

where only one species N_1 has limited carrying capacity. Investigate their stability and sketch the phase-plane trajectories. [Here, K_1, K_2 are carrying capacities; r_1, r_2 are linear birth rates of the populations N_1 and N_2 respectively; b_{12}, b_{21} measure the competitive effects of N_2 on N_1 and N_1 on N_2 respectively.] $4 + 3\frac{1}{2} = 7\frac{1}{2}$

(Turn Over)

(b) What is Routh-Hurwitz criterion? Explain with reference to multiple species communities. $2+5\frac{1}{2}=7\frac{1}{2}$

(c) Discuss bifurcation and limit cycle with respect to any biological model. $7\frac{1}{2}$

UNIT—IV

4. Answer any *two* of the following questions : $7\frac{1}{2}\times2=15$

(a) Write a short note on any *one* of the following :

(i) One-species model with diffusion

(ii) Two-species model with diffusion

(b) For a blood vessel of constant radius R, length L and driving force $P = p_1 - p_2$, show that the average velocity of the flow is equal to half of the maximum velocity and the resistance is proportional to $\dfrac{L}{R^4}$.

(c) Consider arterial blood viscosity $\mu = 0.027$ poise.

If the length of the artery is 2 cm, radius 8×10^{-3} cm and $P = p_1 - p_2 = 4\times10^3$ dynes/cm^2, then find—

(i) $q_z(r)$ and the maximum peak velocity of blood;

(Continued)

(ii) the shear stress at the wall.

(Here q_z denotes velocity along z-axis, p_1 and p_2 denote pressure at two ends of the artery.)

UNIT—V

5. Answer any *two* of the following questions : $10\times2=20$

(a) Let D & d and W & w respectively denote allele for tall & dwarf and round & wrinkled seeds of peas. Find the outcome of the product $DdWw \times ddWw$ using Punneet square or using probability. Also find the probability that the progeny of $DdWw \times ddWw$ is dwarf with round seeds. $6+4=10$

(b) Explain in detail the Hardy-Weinberg equilibrium, mentioning the assumptions considered for the equilibrium.

(c) Compare and contrast stage structure model with age structure model.

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5 SEM TDC DSE MTH (CBCS)
2.1/2.2/2.3/2.4 (H)