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3 SEM TDC MTMH (CBCS) C 5

2024

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C–5

(Theory of Real Functions)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Define cluster point of a set. 1

(b) Write the sequential criterion for limits. 2

(c) Using the definition of limit, evaluate

$$\lim_{x\to 3}\frac{2x+3}{4x-9}$$ 3

Or

Prove that $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist in the set of real numbers.

2. *(a)* What do you mean by one-sided limits of a function f at a point C? 1

(b) Discuss the kind of discontinuity of the function

$$f(x) = \begin{cases} \dfrac{x-|x|}{x}, & \text{when } x \neq 0 \\ 2, & \text{when } x = 0 \end{cases}$$ 2

(c) State and prove squeeze theorem. 3

Or

Apply squeeze theorem to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

(d) Show that the function $f(x)$ defined on $\mathbb{R}$, the set of real numbers, by

$$f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ -x, & \text{when } x \text{ is rational} \end{cases}$$

is continuous only at $x = 0$. 4

Or

Let $A, B \in \mathbb{R}$, let $f : A \to \mathbb{R}$ be continuous on A and let $g : B \to \mathbb{R}$ be continuous on B. If $f(A) \subseteq B$, then prove that the composite function $g \circ f : A \to \mathbb{R}$ is continuous on A.

3. *(a)* State the preservation of intervals theorem. 1

(b) State and prove location of roots theorem. 4

(c) Prove that if a function is continuous in a closed interval, then it is bounded therein. 4

4. *(a)* A function continuous on a closed interval may not be uniformly continuous on that interval. State true or false. 1

(b) Prove that $\sin x$ is uniformly continuous on $[0, \infty]$. 4

Or

Prove that

$$f(x) = \sin\frac{1}{x}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

is not uniformly continuous on $[0, \infty]$.

5. *(a)* Continuity is the sufficient condition for derivability of a function at a point. State true or false. 1

(b) Write the statement of interior extremum theorem. 2

(c) Prove that if $f: I \to \mathbb{R}$ has a derivative at $C \in I$, then f is continuous at C. 3

Or

Examine the differentiability of the function $|x|$ at $x = 0$.

6. *(a)* A convex function on an open interval is necessarily continuous. State true or false. 1

(b) Give an example to show that a convex function need not be differentiable at every point of its domain. 2

(c) Write the geometrical interpretation of Rolle's theorem. 3

Or

Show that the Rolle's theorem is not applicable to the function $f(x) = \tan x$ in the interval $(0, \pi)$.

(d) State and prove Darboux's theorem. 4

7. *(a)* If $\phi'(x) = \psi'(x)$ in an interval, then prove that $\phi(x)$ and $\psi(x)$ differ by a constant in that interval. 2

(b) Prove that

$$\lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a)$$

provided $f''(x)$ is continuous. 3

Or

If $f(x)$ and $\phi(x)$ are continuous in $a \le x \le b$ and differentiable in $a < x < b$ such that $f'(x)$ and $\phi'(x)$ never vanish for the same value of x, then show that

$$\frac{f(\xi) - f(a)}{\phi(b) - \phi(\xi)} = \frac{f'(\xi)}{\phi'(\xi)}$$

where $a < \xi < b$.

(c) Show that $\log(1+x)'$ lies between $x - \frac{x^2}{2}$ and $x - \frac{x^2}{2(1+x)}$, $\forall\, x > 0$. 4

8. *(a)* Write the condition of validity of expansion of e^x in powers of x in infinite series. 1

(b) If $f(x) = x^2$, $\phi(x) = x$, then find a value of ξ in terms of a and b in Cauchy's mean value theorem. 2

(c) Show that the Cauchy's remainder after n terms in the expansion of $\log(1+x)$ in powers of x is

$$(-1)^{n-1}\frac{x^n}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}, \; 0<\theta<1$$ 4

(d) State and prove Cauchy's mean value theorem. 5

Or

Deduce Taylor's theorem from Cauchy's mean value theorem.

9. *(a)* What do you mean by a relative extremum of a function at a point? 1

(b) Show that $x^3 - 6x^2 + 12x - 3$ is neither a maximum nor a minimum when $x = 2$. 2

(c) Give an example of a convex function which is not differentiable at a point. 2

(d) Expand the following functions in powers of x in infinite series stating the conditions under which the expansion is valid (any *two*) : 4×2=8

(i) $\sin x$

(ii) $(1+x)^n$

(iii) $\frac{1}{1+x}$

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