3 SEM TDC MTMH (CBCS) C 6

2024

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C–6

(Group Theory–I)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Give an example of a group of order 37. 1

(b) Is $G = \{0, 1, 2, 3\}$ a group under multiplication modulo 4? Justify your answer. 1

(c) Give two reasons why the set of odd integers under ordinary addition is not a group. 2

(d) Show that the inverse of each element in a group is unique. 2

(e) Consider the elements

$$A=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \text{ and } B=\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}$$

f[illegible] the group $SL(2, \mathbb{R})$. Find $O(A)$, $O(B)$ [illegible] $O(AB)$. 1+1+2=4

Or

Construct a complete Cayley table for the group D_4. 4

(f) If $S=\{1, 2, 3\}$, write the elements of S_3. Show that S_3 is a non-Abelian group with respect to composite of permutations as the composition. 5

2. *(a)* Let G be a group of non-zero real numbers under multiplication. Is

$$K=\{x\in G \mid x=1 \text{ or } x \text{ is irrational}\}$$

a subgroup of G? Justify your answer. 1

(b) Show that $2\mathbb{Z}\cup 3\mathbb{Z}$ is not a subgroup of $\mathbb{Z}$ under ordinary addition. 2

(c) If H is a subgroup of G, then show that $H^{-1}=H$. Show with a counterexample that the converse is not true. 4

(d) Let G be a group and H be a finite non-empty subset of G. Prove that H is a subgroup of G if and only if $ab\in H$, $\forall a, b\in H$. 4

Or

Let H and K be subgroups of G. Show that HK is a subgroup of G if and only if $HK=KH$.

(e) Let G be a group and

$$z(G)=\{z\in G \mid zx=xz, \forall x\in G\}$$

Prove that $z(G)$ is a subgroup of G. Show that G is Abelian if and only if $z(G)=G$. 4

3. *(a)* Write the number of generators of an infinite cyclic group. 1

(b) State true or false with justification : 1

Every group of order 17 is Abelian.

(c) Find the order of

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$ 2

(d) Give an example of a finite Abelian group which is not cyclic. 2

(e) Show that the set A_n of all even permutations of the symmetric group S_n is itself a group under the product of permutations. 4

(f) Prove that every group of prime order is cyclic. 5

(g) State and prove Lagrange's theorem. 5

Or

Let G be a cyclic group generated by an element of order n. If $m < n$ and m, n are relatively prime, prove that a^m is also a generator of G.

4. (a) State true or false : 1

A group of order 31 is simple.

(b) If G is a finite group and N is a normal subgroup of G, prove that

$$O\left(\frac{G}{N}\right) = \frac{O(G)}{O(N)}$$ 2

(c) Prove that the intersection of any two normal subgroups of a group is a normal subgroup. 3

(d) If G is a group, $z(G)$ is the centre of G and $\frac{G}{z(G)}$ is cyclic, show that G is Abelian. 4

(e) Define external direct product of two groups. If H and K are any two groups, then prove that $H \times K$ is Abelian if and only if both H and K are Abelian. 1+4=5

Or

State and prove Cauchy's theorem for finite Abelian groups. 5

5. (a) Define kernel of a group homomorphism. 1

(b) If $f : G \to G'$ is a group homomorphism, then prove that

(i) $f(e) = e'$,

(ii) $f(a^n) = [f(a)]^n$, n is an integer, $a \in G$, where e and e' are identity elements of G and G', respectively. 3

(c) Let G be any group and $\phi : G \to G$ be defined by $\phi(x) = gxg^{-1}$, where g is a fixed element of G. Prove that ϕ is a homomorphism. Also determine $\ker\phi$. 3

Or

Find the regular permutation group isomorphic to the multiplicative group $G = \{1, \omega, \omega^2\}$ of cube roots of unity.

(d) If f is a homomorphism of a group G into a group G' with kernel K, then show that K is a normal subgroup of G. 3

(e) If $f : G \to G'$ is an onto homomorphism with kernel K, then prove that

$$G' \cong \frac{G}{K}$$ 5

Or

State and prove Cayley's theorem.

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