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4 SEM TDC MTMH (CBCS) C 10

2025

(May/June)

MATHEMATICS

(Core)

Paper : C–10

(Ring Theory and Linear Algebra—I)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Give an example of a ring that has no unity. 1

(b) Show that the ring

$$M_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

under matrix addition and multiplication is not an integral domain. 3

(c) Prove that every non-zero element of $\mathbb{Z}_n$ is a unit or a zero divisor. 3

Or

Let p be a prime. Show that $\mathbb{Z}_p$ is a field.

(d) Answer any *two* of the following : 4×2=8

(i) Prove that the characteristic of an integral domain is 0 or a prime.

(ii) Let R be a commutative ring with unity. Let I be an ideal of R. Show that $\frac{R}{I}$ is an integral domain if and only if I is a prime ideal.

(iii) Let R be a ring and S be a non-empty subset of R. If S is closed under subtraction and multiplication, then show that S is a subring of R.

(e) Show that the ideal $\langle x \rangle$ is a prime in $\mathbb{Z}[x]$ but not maximal in $\mathbb{Z}[x]$. 4+1=5

Or

Let R be a commutative ring with unity and I be an ideal of R. Show that I is maximal if and only if $\frac{R}{I}$ is a field. 5

2. (a) Define ring homomorphism. Give an example. 1+1=2

(b) Let R and S be two rings and $f : R \to S$ be a ring homomorphism. Prove that $f(-a) = -f(a),\ \forall a \in R$. 2

(c) Let $f : R \to S$ be a ring homomorphism from ring R to ring S. Show that the set

$$\ker f = \{x \in R : f(x) = 0\}$$

is an ideal of R. 3

(d) Answer any *two* of the following : 4×2=8

(i) Determine all ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$.

(ii) Let R be ring with unity e. Show that the map $f : \mathbb{Z} \to R$ defined by

$$f(n) = ne \quad \forall n \in \mathbb{Z}$$

is a ring homomorphism.

(iii) Let R and S be two rings and $f : R \to S$ be a ring homomorphism. Show that

$$\frac{R}{\ker f} \approx f(R)$$

3. *(a)* Let L be a line in $\mathbb{R}^2$, and the origin is not on L. Is L a subspace of $\mathbb{R}^2$? 1

(b) Show that the set

$$\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}4\\2\\6\end{pmatrix}\right\}$$

is not linearly independent in $\mathbb{R}^3$. 2

(c) Let H and K be two subspaces of a vector space V. Show that the sum

$$H + K = \{h + k : h \in H, k \in K\}$$

is a subspace of V. 3

(d) Let

$$A = \begin{pmatrix}1 & 2 & 4\\0 & 1 & 2\\-1 & 3 & 6\end{pmatrix}$$

Find a basis for col A. 3

(e) Let V be a finite-dimensional vector space and H be a subspace of V. Prove that $\dim H \leq \dim V$. 6

Or

Let V be a vector space and $S = \{v_1, \ldots, v_p\} \subseteq V$ and $H = \text{span}\{v_1, \ldots, v_p\}$

Let one vector in S say v_k be a linear combination of the remaining vectors in S. Show that the set $S - \{v_k\}$ spans H.

4. *(a)* Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that

$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \ \forall \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$$

Find the image of $\begin{pmatrix}1\\-1\end{pmatrix}$ under T. 1

(b) Show that the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\-z\end{pmatrix}, \ \forall \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3$$

is a linear map. 2

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a map defined by

$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\x-y\end{pmatrix}, \ \forall \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$$

Find ker T and range of T. 1+1=2

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map that rotates each point in $\mathbb{R}^2$ about the origin through an angle θ, with counter-clockwise rotation for a positive angle. Find the standard matrix for T. 3

(e) Answer any *three* of the following : 4×3=12

(i) Let $T : V \to W$ be a linear map from finite-dimensional vector space V to W. Show that $\dim V = \text{rank } T + \text{nullity } T$.

(ii) Consider the bases

$$B = \left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right\} \text{ and } C = \left\{ \begin{pmatrix} -7 \\ 9 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \end{pmatrix} \right\}$$

for $\mathbb{R}^2$. Find the change of coordinate matrix from B to C.

(iii) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + y \\ 5x + 7y \\ x + 3y \end{pmatrix}, \ \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

Find the standard matrix for T.

(iv) Define a basis for a vector space V. Let $B = \{b_1, \ldots, b_n\}$ be a basis for V. Let $x \in V$. Show that $\exists$ unique scalars $c_1, \ldots, c_n$ such that

$$x = c_1 b_1 + \ldots + c_n b_n$$

(f) Answer any *two* of the following : 5×2=10

(i) Let V and W be two vector spaces and $T : V \to W$ be a linear map. Show that T is one-one if and only if $\ker T = \{0\}$.

(ii) Let V and W be vector spaces and let $\dim V = n, \dim W = m$. Let B and C be ordered bases for V and W respectively. Let $T : V \to W$ be a linear map. Find the matrix for T relative to B and C.

(iii) Let V and W be vector spaces and $T : V \to W$ be a linear map. If T is invertible, then show that the inverse function $T^{-1} : W \to V$ is a linear map.

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