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**5 SEM TDC MTMH (CBCS) C 12**

**2025**

( Nov/Dec )

MATHEMATICS

( Core )

Paper : C–12

**( Group Theory—II )**

*Full Marks* : 80
*Pass Marks* : 32

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions.*

**1.** *(a)* Define an automorphism of a group $G$. 1

*(b)* State True or False : 1
Characteristic subgroups are normal.

*(c)* Show that the identity mapping of a group is an automorphism. 2

*(d)* Give an example of an inner automorphism of $S_3$. 2

*(e)* Find the automorphism group of $\mathbb{Z}_8$. 4

(f) Let $G$ be a group. Show that the set of all inner automorphisms of $G$ forms a normal subgroup of $\mathrm{Aut}(G)$. 4

(g) Determine the automorphism group of $\mathbb{Z}_{10}$ and its order. 5

*Or*

Show that the commutator subgroup $G'$ of a group $G$ is normal in $G$.

(h) Let $G$ be a group and let $G'$ be the commutator subgroup of $G$. Show that $G/G'$ is abelian. 5

*Or*

If $H$ is a normal subgroup of $G$, then prove that $G/H$ is abelian if and only if $G' \subset H$.

(i) Determine all automorphisms of $\mathbb{Z}$ and prove that $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$. 6

*Or*

Let $G$ be a finite cyclic group of order $n$. Prove that $\mathrm{Aut}(G) \cong U(n)$, the multiplicative group of units modulo $n$.

2. (a) State the definition of external direct product of two groups. 1

(b) What is the order of the group of units $U(9)$? 1





(c) Give an example of two non-isomorphic groups of order 8. 2

(d) State the fundamental theorem of finite Abelian groups. 2

(e) Prove that if $G_1$ and $G_2$ are finite cyclic groups of orders $m$ and $n$ with $\gcd(m, n) = 1$, then $G_1 \times G_2$ is cyclic of order $mn$. 4

(f) Show that $U(15)$ is isomorphic to $U(3) \times U(5)$. 5

(g) Let $G = U(96)$. Express $G$ as an external and internal direct products of cyclic groups. 5

*Or*

Find all Abelian groups (up to isomorphism) of order 360.

3. (a) State Cauchy's theorem for finite groups. 1

(b) Define a $p$-group. 1

(c) Let $G$ is an Abelian group and $a \in G$. Find the conjugacy class of $a$. 2

(d) Give an example of two conjugate elements in $S_4$. 2

(e) Prove that in a $p$-group, the center $Z(G)$ is non-trivial. 4

(f) Let $G$ be a group of order 21. Using Sylow's theorems, determine the number of Sylow 7-subgroups and Sylow 3-subgroups. 4

(g) Prove that any group of order 45 is cyclic. 5

*Or*

Prove that a group of order $p^2$, where $p$ is a prime, is abelian.

(h) State and prove the class equation for a group $G$ acting on itself by conjugation. 5

*Or*

Show that in $S_n$, two permutations are conjugate if and only if they have the same cycle type.

(i) Prove that $A_5$ is simple. 6

*Or*

Prove that no group of order 30 is simple.

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