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3 SEM TDC MTMH (CBCS) C 6

2025

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C-6

(Group Theory—I)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

1. (a) Find the inverse of the element

$$\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$$

in $GL(2, \mathbb{Z}_{11})$.

1

(2)

- (b) State True or False : 1
The set $\{0, 1, 2, 3\}$ is not a group under multiplication modulo 4.
- (c) Prove that in a group G , there is only one identity element. 3
- (d) Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1} \forall a, b \in G$. 4
- (e) Describe the dihedral group D_4 . Show that it is a group together with the operation composition. Is it an Abelian group? Justify. 6

Or

Describe the group of symmetries of a rectangle. Show that it is a group together with the operation composition. Is it Abelian? Justify.

2. (a) Give an example of a non-Abelian group having finite order. 1

(3)

- (b) Find the order of all the elements of $U(10)$. 2
- (c) Let G be an Abelian group with identity e . Show that $H = \{x \in G | x^2 = e\}$ is a subgroup of G . 3
- (d) Define centre, $z(G)$ of a group G . Show that it is a subgroup of G . 1+3=4
- (e) Let H be a non-empty finite subset of a group G and H is closed under the operation of G . Show that H is a subgroup of G . 5

Or

Let

$$G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$$

under addition. Let

$$H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G \mid a+b+c+d=0 \right\}$$

Prove that H is a subgroup of G . What if 0 is replaced by 1?

(4)

3. (a) State True or False : 1
 $U(8)$ is a cyclic group.
- (b) Show that $\mathbb{Z}_{10} = \langle 3 \rangle$. 2
- (c) Find all the generators of $\mathbb{Z}_6$. 2
- (d) Let
$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix} \text{ and}$$
$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{bmatrix}$$

Compute α^{-1} , $\alpha\beta$ and $\beta\alpha$. 3
- (e) Let G be a finite group and let $a \in G$.
Then prove that $a^{|G|} = e$. 3
- (f) Let G be a group and H be a subgroup
of G . Let $a \in G$. Prove that $aH = H$ if
and only if $a \in H$. 4
- (g) State and prove Lagrange's theorem
for finite group. 1+4=5

(5)

Or

Show that every permutation of a finite
set can be written as a cycle or as
a product of disjoint cycles. 5

4. (a) Define normal subgroup. 1
- (b) Find the order of each element in
 $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. 2
- (c) Prove that $SL(2, \mathbb{R})$ is a normal
subgroup of $GL(2, \mathbb{R})$. 3
- (d) Determine the number of elements of
order 5 in $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$. 4
- (e) Let G and H be finite cyclic groups.
Then prove that $G \oplus H$ is cyclic if
and only if $|G|$ and $|H|$ are relatively
prime. 5

Or

Let G be a group and let $z(G)$ be the
centre of G . Prove that if $G/z(G)$ is
cyclic, then G is Abelian.

(6)

5. (a) Define kernel of a homomorphism. 1

(b) State True or False : 1

The kernel of an isomorphism is the identity.

(c) Let ϕ be a homomorphism from a group G to a group $\bar{G}$ and let H be a subgroup of G . Then prove the following : 2+2=4

(i) $\phi(H)$ is a subgroup of $\bar{G}$

(ii) If H is Abelian, then $\phi(H)$ is Abelian

(d) Let ϕ be a group homomorphism from a group G to a group $\bar{G}$. Then show that $\ker \phi$ is a normal subgroup of G . 4

(e) Let ϕ be a group homomorphism from G to $\bar{G}$. Then prove that $G / \ker \phi \approx \phi(G)$. 5

Or

Let G be a group of permutations. For each σ in G , define

$$\text{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

(7)

Prove that sgn is a homomorphism from G to the multiplicative group $\{+1, -1\}$.

What is the kernel? 1+4=5
