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1 SEM TDC MTMH (CBCS) C 2

2019

(December)

MATHEMATICS

(Core)

Paper : C–2

(Algebra)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* State the complex number $-1+i$ in the polar form. 1

(b) Show that the n numbers of n th root of unity form a geometric progression indicating the common ratio. 2

(c) Find the values of $(-16)^{\frac{1}{4}}$. 3

(d) Writing $\cos\theta + i\sin\theta$ as $\text{cis}\theta$, if $x = \text{cis}\alpha$, $y = \text{cis}\beta$, $z = \text{cis}\gamma$ and $xyz = x+y+z$, show that

$$1+\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=0$$ 4

2. (a) Give an example of the well-ordering property of positive integers. 1

(b) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be two functions. Consider the composite of f and g. Following conclusions are drawn :

I. fg is the composite of f and g.

II. Range of f is contained in the domain of g.

Choose the correct answer from the following : 1

(i) Both the statements I and II are true

(ii) I is true and II is false

(iii) I is false and II is true

(iv) Both the statements I and II are false

(c) Consider the functions $f: \mathbb{Z} \to \mathbb{R}$ defined as $f(x) = 2x$ and $g: \mathbb{N} \to \mathbb{R}$ defined as $g(x) = \sqrt{x}$. Find the composites gf and fg, if they exist. Justify your answer in each case. 2

(d) Show that the relation 'congruence modulo m' ($\equiv$) over the set of positive integers is an equivalence relation. 3

(e) Let $f: X \to Y$ be invertible. Show that f is a bijection. Show that $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x + 1$ is a bijection and find its inverse. 3+2+1=6

(f) Let $b > 0$ be an integer and a be any integer. Show that there exist unique integers q and r such that $a = bq + r$, where $0 \le r < b$. 4+2=6

(g) What is Euclidean algorithm? Let $a, b \in \mathbb{Z}$ and either $a \ne 0$ or $b \ne 0$. Show that there exists greatest common divisor d of a and b such that $d = ax + by$ for some integers x and y and d is uniquely determined by a and b. 1+5=6

Or

Show that $an \equiv bn \pmod{m} \Leftrightarrow a \equiv b \left(\bmod \frac{m}{d} \right)$, where $(m, n) = d$.

3. (a) Define linear combination of the vectors $v_1, \cdots, v_p$ in $\mathbb{R}^n$. 1

(b) Give an example of a 3×5 matrix in the row reduced echelon form. 1

(c) A linear system of equations in five variables has been reduced to the associated system

$$x_1 + 6x_2 + 3x_4 = 0;\ x_3 - 4x_4 = 5;\ x_5 = 7$$

with reference to the reduced augmented matrix

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Indicate the basic variables and the free variables. 2

(d) A vector equation $x_1 v_1 + \cdots + x_p v_p = 0$ where each $v_i \in \mathbb{R}^n$; $1 \le i \le p$ and each x_i; $1 \le i \le p$ is a scalar, has the trivial solution. State the consequences with reference to x_i's and v_i's separately. 1+1=2

(e) Define span $\{v_1, \cdots, v_p\}$, where $v_1, \cdots, v_p \in \mathbb{R}^n$. Justify whether $0 \in \text{span}\,\{v_1, \cdots, v_p\}$ or not. Determine, for what value(s) of h, $w = \begin{bmatrix} 3 \\ 1 \\ h \end{bmatrix}$ is in span $\{v_1, v_2, v_3\}$, where $v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. 5

Or

Let A be an $m \times n$ matrix, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. When does the equation $Ax = b$ have a solution? Further for $u, v \in \mathbb{R}^n$ and a scalar c show that—

(i) $A(u+v) = Au + Av$;

(ii) $A(cu) = cAu$.

(f) Describe all the solution of $Ax = b$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix},\ b = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}$$

by—

(i) row reducing the augmented matrix $[A \cdot b]$ to echelon form;

(ii) transforming the above to row reduced echelon form;

(iii) giving the solution in the form $x = p + tv,\ t \in \mathbb{R}$. 2+2+1=5

(g) Prove that an indexed set of two or more vectors $S = \{v_1, \cdots, v_p\}$ is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. 4

Or

Determine a linear dependence relation among the vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

4. (a) Let $T: \mathbb{R}^5 \to \mathbb{R}^2$ and $T(x) = Ax$ for some matrix A and for each $x \in \mathbb{R}^5$. How many rows and columns are there in A? 1

(b) Define the column space of a matrix A. 1

(c) Show that the null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. 2

(d) Show that $u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and state the corresponding eigenvalue. 2

(e) Determine the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$. 2

(f) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then show that T is one-to-one if and only if the equation $T(x) = 0$ has the trivial solution. 4

(g) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Show that there exists a unique matrix A such that $T(x) = Ax \ \forall \ x \in \mathbb{R}^n$. 4

Or

Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be linear and given

$$T(e_1) = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \ T(e_2) = \begin{bmatrix} -5 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \text{ where } e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find a formula for the image of an arbitrary x in $\mathbb{R}^2$.

(h) Row reduce the augmented matrix $[A \quad I]$, where $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ and I, the identify matrix so that $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Verify that $AA^{-1} = I$. 3+2=5

(i) Determine the rank of

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

by row reducing it to echelon form. 4

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