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3 SEM TDC MTMH (CBCS) C 6

2020

(Held in April–May, 2021)

MATHEMATICS

(Core)

Paper : C–6

(Group Theory—I)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Write the order of the group P_n. 1

(b) Write the identity element of the group Z of all integers under the operation defined by $a \quad b \quad a \quad b \quad 1, \quad a, b \quad Z$. 1

(c) If G is a group, then prove that

$$O(x) \quad O(x^{\ 1}), \quad x \quad G$$ 2

(d) Define dihedral group. 2

(e) Let G be a group and $a, b \quad G$. Then prove that the equation $ax \quad b$ has a unique solution in G. 4

Or

If G is a group, then prove that—

(i) identity element of G is unique;

(ii) inverse of every element of G is unique;

(iii) $(x^{\ 1})^{\ 1} \quad x, \quad x \quad G$.

(f) Show that the four permutations

$$I, (ab), (cd), (ab)(cd)$$

on four symbols a, b, c and d form a finite Abelian group under the operation multiplication of permutations. 5

2. *(a)* Write the class equation of the group G. 1

(b) Give an example to show that the union of two subgroups of a group G is not necessarily a subgroup of G. 2

(c) If H and K are two subgroups of a group G, then prove that HK is a subgroup of G iff $HK \quad KH$. 4

Or

Show that a non-empty subset H of a group G is a subgroup of G iff $HH^{1} \quad H$.

(d) Prove that the centre Z of a group G is a normal subgroup of G. 4

(e) Let G be a finite group and Z be the centre of G, then prove that the class equation of G can be written as

$$O(G) \quad O(Z) \quad _{a \; Z} \frac{O(G)}{O[N(a)]}$$

where the summation runs over one element a in each conjugate class containing more than one element. 4

3. *(a)* Write the number of generators in a cyclic group of order 15. 1

(b) What do you mean by index of a set S in a group G? 1

(c) Examine whether the following permutation is even or odd : 2

$$f \quad (1 \;\; 3 \;\; 2 \;\; 5 \;\; 6 \;\; 7)$$

(d) Find the order of

$$f \quad \begin{matrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{matrix}$$

2

(e) Prove that any two right cosets are either identical or disjoint. 4

(f) Prove that every subgroup of a cyclic group is cyclic. 5

Or

Prove that every group of prime order is cyclic.

(g) State and prove Lagrange's theorem. 5

4. *(a)* State True or False : 1

Any group of prime order is simple.

(b) Define external direct product of groups. 2

(c) Show that every subgroup of an Abelian group is normal. 3

(d) Prove that every quotient group of a cyclic group is cyclic. 4

(e) If G_1 and G_2 are groups, then prove that $G_1 \quad \{e_2\}$ and $\{e_1\} \quad G_2$ of $G_1 \quad G_2$ are normal subgroup of $G_1 \quad G_2$ isomorphic to G_1 and G_2 respectively. 5

Or

Let G be a finite Abelian group such that its order $O(G)$ is divisible by a prime p, then prove that G has at least one element of order p.

5. *(a)* Let f be a homomorphism from a group G into a group G . Then prove that—

(i) $f(e) \quad e$, where e, e are identities of G and G respectively;

(ii) $f(a^n) \quad [f(a)]^n, \quad a \quad G.$ 2

(b) Let $(Z, \quad)$ and $(E, \quad)$ be the group of integers and the group of even integers respectively. Then show that the mapping $f : Z \quad E$ defined by

$$f(x) \quad 2x, \quad x \quad Z$$

is an isomorphism. 3

(c) Prove that every finite group G is isomorphic to a permutation group. 5

(d) If $f : G \quad G$ is an onto homomorphism with kernel K, then prove that

$$G \cong G / K$$ 5

Or

Let H be a normal subgroup of a group G and K be a subgroup of G. Then prove that

$$\frac{HK}{H} \cong \frac{K}{H \quad K}$$

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