SOME ASPECTS OF TOPOLOGY

A Dissertation Submitted to the Dibrugarh University as a partial fulfilment of requirements for the Degree of Master of Philosophy in Mathematics.

BY

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Date 14/10/95

TO WHOM IT MAY CONCERN

This is to certify that the dissertation entitled "**SOME ASPECTS OF TOPOLOGY**" submitted by Sri Lok Bikash Gogoi, M.Sc., for the award of the Degree of Master of Philosophy in Mathematics to Dibrugarh University is a record of bonafide research work done by him under my supervision and guidance. He has fulfilled all the requirements for submitting the dissertation for the degree of Master of Philosophy.

This dissertation have not been submitted to any other University or Institute for award of any other degree or diploma.

14/10/95

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Lok Bikash Gogoi

Dated, Dibrugarh
The 21th September,1995

(LOK BIKASH GOGOI)

CONTENTS

CHAPTER - I

INTRODUCTION :

1.1 Priliminaries :

The word "Topology" is derived from the Greek word Τόπος, which means "Place", "Position" or "Space". Accordingly, topology is the science of space, it analyses the space concept and investigate the properties of general spaces. It is therefore a subdiscipline of geometry. This does not keep it from being in close and truithtful relation to analysis and algebra. It provides analysis with geometric foundation : it receives, on the other hand, essential stimuli from analysis (c.f. functional analysis). From algebra as the fundamental basic and auxiliary discipline of mathematics, it takes essential helping material (e.g. linear algebra, group and module theory) and gives it, in turn, important new result (e.g. homological algebra). However, the proper goal of topology it always the acquisiton of geometric knowledge.

The dictionary meaning of topology is "use the place name as an aid to memory" or "by associating things with particular town or region".

It is Riemann who should be considered as the creator of topology, as of so many other braches of modern mathematics. He was the first to attempt to formulate the notion of a topological space ; he conceived the idea of an autonomous theory of such spaces ; he defined invariants (the "Betti Numbers") which were to play a pre-eminent part in the later development of topology ; and he was the first to apply topology to analysis (periods of abelian integrals). Now-a-days the word "topology" is being commonly used and getting popularity day by day in the field of modern mathematics. The most of the development of topology took place since 1900. In the recent times, topology has been firmly established as one of the basic disciplines of pure mathematics. It has also greatly stimulated the growth of abstract algebra. As things stands today, much of the modern pure mathematics must remain a closed book to the person who does not acquire a working knowledge of at least the elements of topology.

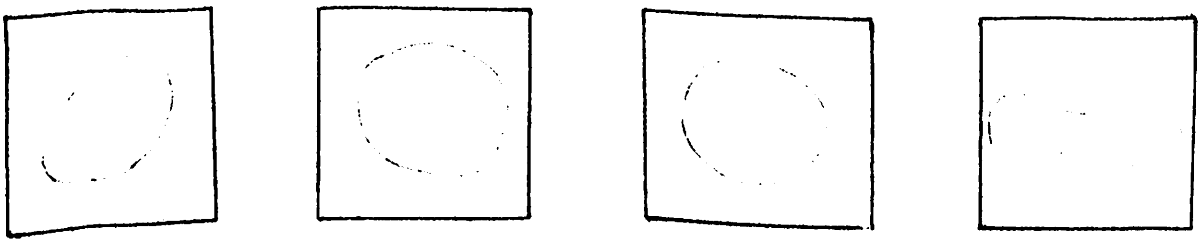
There are many domains in the broad field of topology of which of the following are only few, the homology and cohomology theory of the examples, and of more general spaces as well, the dimension theory, the theory of differentiable and Riemannian manifolds and Lie groups, the theory of continuous curves, the theory of Branch and Hilbert spaces and their operators and the Branch algebras and abstract harmonic analysis on locally compact groups.

There are two types of topology, algebraic topology and point set topology. The algebraic topology in--cludes the use of algebraic methods, while point set topology is the study of sets as accumulations of points and deserving sets in terms of topological properties such as being open, closed, compact, connected, normal, regular and complete etc., including the nets and filters.

Topology is also known as ANALYSIS-CITUS. It is a branch of màthematics treating the properties of a space that are invariant under homeomorphisms. A homeomorphism is a continuous one-to-one tranformation whose inverse is also continuous. It deals with properties of geometrical figures, properties that do not change even though the figures themselves undergo change and such properties of spaces are known as the invariant properties.

For illustration, we take an example - If we have a very elastic or approximately perfectly elastic sheet of rubber and draw some geometrical figures on it, without cutting or tearing the sheet if it is allowed to stretch and bend, then it is remarkable that difference between any two points on unstretched form, may be made larger at will, but the path between the two points remains such that it does not cross itself, whatever mode of stretching or bending is adopted. Similarly by a careful stretching an angle of 20^0 on an unstretched sheet may be changed to become 60^0 . These sizes of the angles betwen any two points have no consideration in topology, but it is the path betwen the two points, which is known as arc and given due consideration in topology.

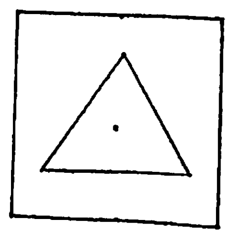
For further illustration, if we stretch a rubber band and bend it within in elastic limits it forms a closed circuit. Formation of a closed circuit is an intrinsic property of the rubber band. Let us have a look at the shapes of a rubber band



(i) (ii) (iii) (iv)

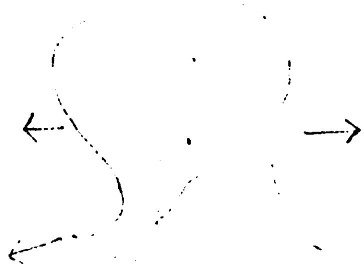
(i), (ii), (iii) and (iv) are some of the shapes obtained when a rubber band is stretched within its elastic limit.

Let us draw a closed surface, say a triangle on a thin sheet of rubber, with a dot inside.

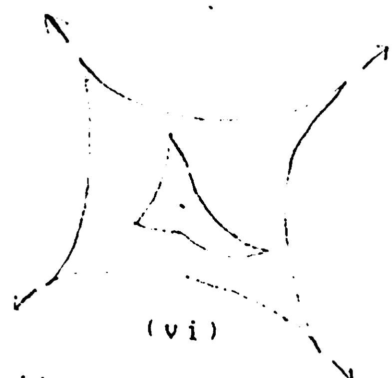


Rubber sheet

Now, let us stretch the rubber sheet in the following ways and twist the rubber sheet.



(v)



(vi)

Triangle on the sheet will undergo transformation indicated on the stretched sheets in the above (v) and (vi) figures.

In this way in whatever manner we stretch there is one common feature in all such transformations of the rubber sheet . In the figures (v) and (vi) the arrows indicated the directions in which the original rubber sheet is stretched.

We observed that the common feature is that the dot inside the triangle remains always inside even after stretching in whatever manner we want to. In other way we say that the location of the dot is an invariant. In other word, containment of dot within the triangle is an invariant.

From the topological point of view, a circle, an ellipse, a polygon and indeed any complete and uncrossed line are equivalent. The curve that does not cross itself is a simple curve, which divides the plane surface into two regions. One region contains all the points inside the curve and the other contains all the points outside the curve.

Therefore, in topology the concept of space is considered to be as genral as possible ; it should comprise everything which in the widest sense of the word deserves the same space. To this concept belong, besides the fundamental basic models (the ordinary three dimensional Euclidean space R^3 and the n-dimensional space R^n, with n=1,2,3,-------- and all subsets of R^n), the infinite dimensional Hilbert space, the non-Euclidean spaces and the spaces of Riemannian Geometry, as well as more general formations, e.g., the 4-dimensional set of lines in R^3, the set of ellipsoids in R^n, the phase spaces in physics, matrix spaces and function spaces, and many other more general spaces which will not be described here. Naturally, it is not a matter here of the particular properties of one or another of these examples, but rather of the characteristic properties common to all these spaces. Since topology strives for the most penetrating analysis possible of the space concept, it has not only mathematical, but also has philosophical characteristics (e.g., concerning the theory of cognition), especially in the fundamental portions. Whereas a much discussed classical philosophical teaching (cf. I. Kant, 1724-1804) asserts that the Euclidean geometry of R^3 is the necessary form of human space perception.

The point of departure and the methods of topology as well as its relations to its neighbour disciplines can be indicated by an especially important examples, viz., the domain of real numbers, which certainly is of fundamental importance for many other portions of mathematics. Real numbers can be added and multiplied, and the laws which addition and multiplication obey can be derived from fewer basic laws, the so called field laws. Algebra investigates these basic laws and their consequences. It considers more general systems which are defined automatically and in which combining operations similar to addition and multiplication with the same or similar basic rules as the axioms are present. Thus, one arrives at the concepts of field, ring, group and others. and the theory of these algebraic structures. Topology is not interested, in the combining operations of the real numbers or their generalisations. It directs its attention more towards those properties which the real number have, say, due to the fact that the numerical sequence 1, 1/2, 1/3, ---------- has the limit zero. It deals with the concepts of neighbourhood proximity or the property of being neighbouring, openness or closeness of sets of real numbers, continuity of real valued functions, and similar concepts. From among these concept, it choose the simplest possible and the least number possible are chosen as axioms. Thus, one arrives at the fundamental concept of a general topologecal space, entirely analogous to the above described procedure in algebra.

The theory of topological space or as it is called point set or general topology has become one of the elementary building blocks underlying diverse branches of mathematics. Its concepts and methods have enriched numerous other fields of mathematics like functional analysis, algebraic and differential geometries and given impetus to their future development. For these reasons topological structures are considered to be one of those few basic structures which gives access to modern mathematical research.

It is difficult to decide the precise time. when the development of topology as a subject in its own right began, problems of topological nature were considered by Euler and Gauss. Frechet's initiated the study of metric spaces in 1906 in his Doctoral thesis [Frechet, M. Sur Quelques points du calcul fonctionnel, Rendiconti dipalermo 22 (1906) 1-74].

1.2. SOME IMPORTANT DEFINITIONS AND RESULTS INCORPORATED INTO THIS DISSERTATION

1.2.1 : Topologies

A topology on a set X is a structure given by a set T of subsets of X having the following three properties called axioms of topology :

(i) The set X and the empty set ϕ are sets of T.

(ii) Every union of sets of T is a set of T.

(iii) Every finite intersection of sets of T is a set of T.

The sets of T are called open sets of the topology defined by T on X.

1.2.2 : Topological spaces

A topological space is a set endowed with a topology. i.e., when a topology T has been defined on a set X, the pair (X,T) is said to be a topological space.

The elements of a topological space are often called points.

1.2.3. : Trivial topologies

(i) Discrete topology : Let X be any set and D be the collection of all subsets of X. Then D is a topology for X called the Discrete topology and the pair (X,D) is called a discrete topological space.

(ii) Indiscrete topology : Let X be any set. Then the collection $I = \{\varphi, X\}$ is always a topology for X called the indiscrete topology. The pair (X, I) is called an indiscrete topological space.

1.2.4. : Comparison of topologies

Let T_1 and T_2 be two topologies for a set X. We say that T_1 is coarser or weaker or smaller than T_2 or that T_2 is finer or stronger or larger than T_1 iff $T_1 \subset T_2$.

If either $T_1 \subset T_2$ or $T_2 \subset T_1$ we say that the topologies T_1 and T_2 are comparable. If $T_1 \not\subset T_2$ and $T_2 \not\subset T_1$, then we say that T_1 and T_2 are not comparable.

For any set X, the indiscrete topology I is the coarsest topology and the discrete topology D is the finest topology.

1.2.5. : Usual topology

Let R denote the set of real numbers. We define a collection T of subsets of R as follows :

A subset U of R is in T iff for an arbitrarily given point $u \in U$, there exists a positive real number δ_u such that a real number x is in U if $|x - u| < \delta_u$. Then this collection T forms a topology in R which is known as usual topology in R and the pair (R, U) is a usual topological space.

1.2.6. : <u>Co-finite or finite complement topology</u>

Let X be any set and let T be the collection of all those subsets of X whose complements are finite together with the empty set, i.e., a subset A of X belongs to T iff A is empty or A' is finite. Then T is a topology for X called the co-finite topology or the finite complement topology.

1.2.7. : <u>Co-countable topology</u>

Let X be a set and let T consists of all those subsets of X whose complements are countable sets together with the empty set. Then T is a topology for X called the Co-countable topology.

1.2.8. : <u>Intersection and Union of topologies</u>

Let X = { a, b, c }. Consider two topologies T_1 and T_2 for X defined as follows :

$T_1 = \{ \phi , \{ a \}, X \}$

$T_2 = \{ \phi , \{ b \}, X \}$

then $T_1 \cup T_2 = \{ \phi , \{a\}, \{b\}, X \}$

which is not a topology, since

$\{a\} \cup \{b\} = \{a, b\} \notin T_1 \cup T_2$.

Thus, union of topologies is not necessarily a topology on X. However, the intersection of any collection of topologies is a topology for X.

1.2.9. : Metric topolgy

Let (X, d) be a metric space and let T_d consists of ϕ and all those subsets G of X having the property that to each $x \in G$, there exists $r > 0$ such that the open sphere S(x, r) is contained in G. Then T_d is a topology for X and is said to be topology induced by the metric d.

1.2.10. : metrizable topological spaces

A metric on a set X is said to be compatible with a topology T on X if the topology defined by this metric coincides with T . A topological space is said to be metrizable if there exists a metric on X compatible with the topology for X .

1.2.11. : Neighbourhoods

Let X be a topological space and A be any subset of X . A neighbourhood of A is any subset of X which contains an open set containing A . The neighbourhood of a subset { x } consisting of a single point is also called neighbourhood of the point x.

For example, if $X = \{a, b, c, p\}$ and T be a topology on X as $T = \{\phi, \{b, c\}, \{a, p\}, \{a, c, p\}, \{c\}, X\}$.

Also $A = \{a, b, p\} \subset X$.

Open set containing p is $\{a, p\}$ which is such that $p \in \{a, p\} \subset \{a, b, p\}$.

Hence A is the neighbourhood of p .

1.2.12. : Properties of neighbourhoods

Ler X be a topological space and for each $x \in X$. let N(x) be the collection of all neighbourhoods of x . Then

$N_0 : \forall x \in X, \; N(x) \neq \phi$

That is, every point x has at least one neighbourhood.

$N_1 : N \in N(x) \Rightarrow x \in N$

That is, every neighbourhood of x contains x.

N_2 : $N \in N(x)$, $M \supset N \Rightarrow M \in N(x)$

That is, every set containing a neighbourhood of x is a neighbourhood of x.

N_3 : $N \in N(x)$, $M \in N(x) \Rightarrow N \cap M \in N(x)$

That is, the intersection of two neighbourhoods of x is a neighbourhood of x.

N_4 : $N \in N(x) \Rightarrow \exists\ M \in N(x)$ such that $M \subset N$ and $M \in N(y)$, $\forall\ y \in M$

That is, if N is a neighbourhood of x , then there exists a neighbourhood M of x which is a subset of N such that M is a neighbourhood of each of its points.

1.2.13. : <u>Fundamental system of neighbourhoods</u>

In a topological space X , a fundamental system of neighbourhoods of a point x (respectively of a subset A of X) is any set E of neighbourhoods of x (respectively A) such that for each neighbourhood V of x (respectively A) there is a neighbourhood $W \in E$ such that $W \subset V$.

If E is a fundamental system of neighbourhoods of a system A of X , then every finite intersection of sets of E contains a set of E .

For example, on the rational line Q, the set of all open intervals containing a point x is a fundamental system of neighbourhoods of this point.

1.2.14. : <u>Base of a topology</u>

Let (X, T) be a topological space. A collection B(x) of subsets of X is known to be a Base for topology T iff

(i) $B(x) \subset T$

(ii) Each member of T can be expressed as the union of the members of B(x).

For example, let

$X = \{ 1, 2, 3 \}$

$T = \{ \phi, \{ 1 \}, \{ 2 \}, \{ 3 \}, \{ 1, 2 \}, \{ 1, 3 \}, \{ 2, 3 \}, X \}$

is a topology on X.

Also $B(x) = \{ \phi, \{ 1, 3 \}, \{ 2, 3 \}, X \}$

Now B(x) is a subset of T and union of members of B(x) i.e., $\phi \cup \{ 1, 3 \} \cup \{ 2, 3 \} \cup X = X$ is a member of T which are the conditions for a set to be a base for a topology. But B(x) is not a topology as $\{ 1, 3 \} \cap \{ 2, 3 \} = \{ 3 \} \notin B(x)$. Therefore B(x) is not a base.

Again let,

$T = \{ \phi , \{1\}, \{3\}, \{1, 3\}, \{3, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}, X \}$ be a topology on X where, $X = \{1, 2, 3, 4, 5\}$.

Also, $B(x) = \{ \phi , \{1\}, \{3\}, \{3, 4\}, \{1, 2, 3\}, X \}$

Here $B(x) \subset T$ and each member of T is the unoin of members of B(x), such as, $\phi = \phi \cup \phi$ etc. Since ϕ is the subset of every set and so is of B(x) and

$\{1\} = \phi \cup \{1\}$; $\{3\} = \phi \cup \{3\}$;

$\{1, 3\} = \{1\} \cup \{3\}$; $\{3, 4\} = \{3\} \cup \{3, 4\}$;

$\{1, 2, 3\} = \{1\} \cup \{3\} \cup \{1, 2, 3\}$;

$\{1, 3, 4\} = \{1\} \cup \{3\} \cup \{3, 4\}$;

$\{1, 2, 3, 4\} = \{1\} \cup \{3\} \cup \{3, 4\} \cup \{1, 2, 3\}$

$X = \{1\} \cup \{3\} \cup \{3, 4\} \cup \{1, 2, 3\} \cup X$

Hence B(x) is a base for T.

If X is a topological space and B(x) is a subset of X. The elements of B(x) are open sets and these are known as basic open sets.

1.2.15. : Local base of a topological space

By a local base or a neighbourhood base, of a space X at a point $x \in X$, we mean a collection B_x of

Neighbourhoods of x in X such that every neighbourhood of x in X contains a member of B_x. The menbers of a given local base B_x of X at the point $x \in X$ is called the basic neighbourhoods of x in X.

Example of Local base

Consider a real line R and any $p \in R$ for each positive real number $\delta > 0$, the set

$$N_\delta(p) = \{ x \in R \mid |x - p| < \delta \}$$

is called the δ - neighbourhood of p in R. Let $\delta_1, \delta_2 \cdots \cdots \delta_i \cdots \cdots$ be any sequence of positive real numbers converging to 0 : then the collection $\{ N_{\delta_i}(p) : i = 1, 2, 3, \text{-------} \}$ is a local base of R at p.

1.2.16. : Subbase of a topological space

Let (X, T) be a topological space. A collection B^* of subsets of X is called a Sub-base for the topology T iff $B^* \subset T$ and finite intersection of members of B^* forms a base for T.

It follows that B^* is a Sub-base for T iff every member of T is the union of finite intersections of members of B^*.

Example of Sub-base

Let $T = \{ \phi , \{a\}, \{a, c\}, \{a, d\}, \{a, c, d\}, X \}$ be a topology on X where $X = \{a, b, c, d\}$.

Then the collection

$B^* = \{ \{a, d\}, \{a, c\} \}$ is a Sub-base for T since the family B of finite intersections of B^* is given by,

$B = \{ \{a\}, \{a, c\}, \{a, d\} \}$

which is a base for T.

1.2.17. : Closed sets in a topological space

In a topological space X, the complements of the open sets of X are closed sets.

For example, let $X = \{a, b, c\}$ and let

$T = \{ \phi , \{a\}, \{b, c\}, X \}$ be a topology on X

Since $\{a\}' = \{b, c\}$

$\{b, c\}' = \{a\}$

it follows that the closed sets of X are

X, $\{b, c\}$, $\{a\}$ and ϕ.

1.2.18 : G_δ & F_σ and Borel sets :-

A subset G of a topological space X is called a G_δ iff it is a countable intersection of open sets and a subset F is called a F_σ iff it is a countable union of closed sets.

Borel sets

The family of Borel sets in a topological space X is the smallest family of sets $\mathcal{G}$ with the following properties :

(i) $\mathcal{G}$ contains the open set.

(ii) Countable intersection of elements of $\mathcal{G}$ belongs to $\mathcal{G}$.

(iii) Complements of elements of $\mathcal{G}$ belongs to $\mathcal{G}$.

1.2.19. : **Door space**

A topological space (X, T) is said to be a Door space iff every subset of X is either open or closed.

Example :

Let T = { ϕ , { b }, { a, b }, { b, c }, X } be a topology on X where x = { a, b, c } . Then the closed sets of X are X, { a, c }, { c }, { a }, ϕ .

Hence all the subsets of X are either closed or open and consequently (X, T) is a door space.

1.2.20. : **Characterisation of a topological space in terms of closed sets**

Let x be a non-empty set and $\mathcal{F}$. a family of subsets of X such that,

F_1 : $\Phi \in \mathcal{F}$, $X \in \mathcal{F}$

F_2 : F_1 , $F_2 \in \mathcal{F} \Rightarrow F_1 \cup F_2 \in \mathcal{F}$

F_3 : $F_\lambda \in \mathcal{F} \; \forall \; \lambda \in \Lambda \Rightarrow \cap \{ F_\lambda : \lambda \in \Lambda \} \in \mathcal{F}$

Then there exists a unique topology for X such that the closed subsets of X are precisely the members of $\mathcal{F}$.

1.2.21. : First countable space

A topological space (X, T) is said to satisfy the first axioms of countability if each point of X possesses a countable local base. Such a topological space is said to be first countable space.

Example : A discrete space (X, D) is first countable. In a discrete space, every subset of x is open. In particular, each singleton set { x }, $x \in X$ is open and so is a neighbourhood of x. Also, every neighbourhood N of x must be a superset of { x }. Hence the collection { { x } } consisting of the singleton neighbourhood { x } of x constitutes a local base at x. But a collection consisting of a single member is countable. Hence there exists a countable base at each point of X.

1.2.22. : <u>Second countable spaces</u>

Let (X, T) be a topological space. The space is said to be second countable or to satisfy the second axiom of countability iff there exixts a countable base for T.

For example, the usual topological space (R, U) is second countable since the set of all open intervals]r, s[where r, s are rational numbers, forms a countable base for U .

1.2.23. : <u>Limit points</u>

Let A be subset of a topological space X. A point $x \in X$ is called a limit point or an accumulation point or a cluster point of A iff every neighbourhood of x contains a point of A other than x .

1.2.24. : <u>Derived set</u>

The set of all limit points of a subset A of X is called the derived set of A and shall be denoted by D(A).

Example : Let X = { a, b, c } and

let T = { φ , { a }, { a, b }, { a, c }, X } be a topology on X and A = { a, c }

Here a is not a limit point of A , since { a } is a neighbourhood of a which contains no point of A other than a. But b is a limit point of A . since open neighbourhood of b are { a, b } and X , each of which contains a point of A other than b. Also c is a limit point of A , since there are only two open neighbourhoods of c , namely, { a, c } and X each of which contains a point of A other than c. Since b , c are limit points of A, we have

$$D(A) = \{ b, c \}$$

1.2.25. : Adherent points

Let A be a subset of a topological space X and let $x \in X$. Then x is called an adherent point (also called contact point) of A iff every neighbourhood of x contains a point of A. The set of all adherent points of A is called the adherence of A and is denoted by AdhA.

1.2.26. : Isolated and Perfect points :

A point x is said to be an isolated point of a subset A of a topological space X if x belongs to A but x is not a limit point of A, i.e., there exists some neighbourhood N of x such that N contains no point of A other than x. A closed set which has no isolated points is said to be a perfect.

It follows that an adherent point is either an isolated point or a limit point of A. For if x is an adherent point of A, then there are following two mutually exclusive possibilities :

(i) Every neighbourhood N of x contains a point of A other than x. In this case x is a limit point of A.

(ii) For x $\in$ A and there is some neighbourhood of x which contains no point of A except x. In this case x is an isolated point.

For example, let,

X = { a, b, c, d, e } and let

T = { ϕ , {b}, {d, e}, {b, d, e}, {a, c, d, e}, X }

Then T is a topology on X. We consider the subset A = { b, c, d }. Then c is a limit point of A, Since the open neighbourhoods of c are { a, c, d, e } and X each of which contains a point of A other than c. But b is not a limit point of A, since { b } is an open neighbourhood of b which contains no point of A other than b. Similarly, the other limit points of A are a, e. Hence

D(A) = { a, c, e }

Isolated points of A are b and d, since b, d belongs to A, but are not limit points of A. The adherent points of A are a, b, c, d, e.

1.2.27. : <u>Closure</u>

Let (X,T) be a topological space and A be a subset of X, i.e., $A \subseteq X$. The closure of A denoted by $\bar{A}$ is defined as the intersection of all closed supersets of A i.e., the intersection of all closed subsets of X which contains A or symbolically,

$$\bar{A} = \bigcap_i F_i$$

where $\{ F_i : i \in I \}$ is the class of all closed subsets of X which contain A.

Example : If X = { 1,2,3,4,5} and

T = { ϕ , X, {1,2}, {1,3}, {2,5}, {1,2,3}, {1,2,5}, {1,2,3,5}, {1}, {2} }

Then T is a topology on X.

Let us consider A = {1,4} $\subset$ X

Closed sets of X are X, ϕ , {3,4,5}, {2,4,5}, {1,3,4}, {4,5}, {3,4}, {4}, {2,3,4,5}, {1,3,4,5}

Therefore, closure of A i.e.,

$\bar{A}$ = Intersection of closed sets which contain A

= X $\cap$ {1,3,4} $\cap$ {1,3,4,5}

= {1,3,4}

1.2.28. : Closure operator and Kuratowski closure axioms

Let (X, T) be a topological space. A closure operator on X is a function C : P(X) --> P(X) satisfying the following four conditions known as Kuratowski closure axioms :

(K_1) $C(\emptyset) = \emptyset$

(K_2) $A \subset C(A)$

(K_3) $C(A \cup B) = C(A) \cup C(B)$

(K_4) $C(C(A)) = C(A)$

Where A and B are any subsets of X.

The topological space (X, T) then called Kuratowski space.

1.2.29. : Interior points

In a topological space X, a point x is said to be an interior point of a subset A of X if A is a neighbourhood of X. The set of interior points of A is called the Interior of A and it is denoted by Int(A) or by A^o .

From the defination of neighbourhood, a point x is interior point of A if there is an open set contained in A which contains x ; it follows that Int(A) is the union of all the open sets contained in A, and hence is the largest open set contained in A.

For example,

If $X = \{1,2,3,4,5\}$ and

$T = \{ \phi, \{1,2\}, \{1,3\}, \{2,5\}, \{1,2,3\}, \{1,2,5\}, \{1,2,3,5\}, \{1\}, \{2\}, X \}$

is a topology on X.

Let us consider $A = \{1,3,4\} \subseteq X$

Union of open sets contained in A are

$\phi \cup \{1,3\} \cup \{1\} = \{1,3\}$

Therefore, $Int(A) = \{1,3\}$

1.2.30. : Interior operator

In a topological space X, the interior operator on X is a function $i : P(X) \longrightarrow P(X)$ satisfying the following conditions, usually known as interior axioms :

I_1 : $i(X) = X$

I_2 : $i(A) \subset A$

I_3 : $i(A \cap B) = i(A) \cap i(B)$

I_4 : $i(i(A)) = i(A)$

Where A and B are subsets of X.

1.2.31. : Exterior points and Exterior of a set

In a topological space X, every point which is interior to the complement of a set $A \subseteq X$ is said to be an exterior point of A. and the set of these points is called the exterior of A in X.

A point $x \in X$ which is an exterior point of A is therefore cahracterized bythe property that x has a neighbourhood which does not meet A.

1.2.32. : <u>Exterior operator</u>

On a topological space X, The exterior operator is a function $e : P(X) \dashrightarrow P(X)$ satisfying the following conditions, usually known as exterior axioms :

E_1 : $e(X) = \emptyset$, $e(\emptyset) = X$

E_2 : $e(A) = A'$

E_3 : $e(A) = e[(e(A))']$

E_4 : $e(A \cup B) = e(A) \cap e(B)$

Where A and B are subsets of X.

1.2.33. : <u>Frontier point and Frontier of a set</u>

In a topological space X, a point x is said to be a frontier point of a set A if x lies in the closure of A and in the closure of complement of A. The set of frontier points of A is called the Frontier of A.

The frontier of A is therefore the set $\bar{A} \cap \bar{A'}$, which is closed.

1.2.34. : <u>Dense sets</u>

In a topological space X, if A, B are subsets of X, then

(i) A is said to be dense in B iff $B \subset \bar{A}$

(ii) A is said to be dense in X or everywhere dense iff $\bar{A} = X$

It follows that A is everywhere dense iff every point of X is an adherent point of A.

(iii) A is said to be nowhere dense or non-dense in X iff $(\bar{A})^{o} = \phi$, that is, iff the interior of the closure of A is empty.

(iv) A is said to be dense-in-itself iff $A \subset D(A)$ i.e., iff A is the subset of derived set of A.

For example,

Let $X = \{ x, y, z \}$ and

$T = \{ \phi, \{ y \}, \{ y, z \}, \{ x, y \}, X \}$

be a topology on X.

If $A = \{ y, z \} \subset X$

Then closed sets of X are X, { x, z }, { x }, { z }, ϕ

Now closure of $A = \overline{A} = \{ x, y, z \} = X$

So A is dense.

1.2.35. : <u>Continuous functions</u>

A mapping f of topological space X into a topological space Y is said to be continuous at a point $x \in X$ if, given any neighbourhood V of $f(x)$ in Y, there is a neighbourhood U of x in X such that $f(U) \subset V$.

A mapping f of a topological space X into a topological space Y is said to be continuous on X or simply continuous at every point of x.

For example, every mapping of a discrete space into a topological space is continuous.

1.2.36. : <u>Open and closed mapp ings</u>

Let X, Y be two topological spaces. A mapping $f : X \longrightarrow Y$ is open (respectively closed) if the image under f of each open (respectively closed) set of X is open (respectively closed) in Y.

Example : If T_1 and T_2 are two topologies on X and Y respectively, where

$X = \{1,2,3\}$

$T_1 = \{ \phi , \{2\}, \{1,3\}, X \}$

$Y = \{ a, b \}$

$T_2 = \{ \phi , \{ a \}, \{ b \}, Y \}$

and the function is

f : (1, b), (2, a), (3, b)

Therefore, $f(\{1,3\}) = \{ b \} \in T_2$

$f(\{2\}) = \{ a \} \in T_2$

Hence the mapping f is open.

1.2.37. : Bicontinuous mappings

Let X, Y be two topological spaces. A mapping $f : X \dashrightarrow Y$ is bicontinuous iff f is open and continuous.

For example, if $T_1 = \{ \phi , \{a\}, \{b,c\}, X \}$ is a topology on $X = \{a,b,c\}$ and $T_2 = \{ \phi , \{z\}, \{x,y\}, Y \}$ is a topology on $Y = \{x,y,z\}$ then the mapping

f : (a,z), (b,x), (c,y) is a bicontinuous, as f is continuous as well as open.

f is continuous, since

$f^{-1}[\phi] = \phi$, $f^{-1}[Y] = X$

$f^{-1}[\{z\}] = \{a\} \in T_1$

$f^{-1}[\{x, y\}] = \{b, c\} \in T_1$

i.e., the inverse image of every T_2 -open set is T_1 -open.

f is open since,

$f[\phi] = \Phi$, $f[\{a\}] = \{z\}$

$f[\{b, c\}] = \{x, y\}$, $f(X) = Y$

i.e., the image of every T_1-open set is T_2-open.

1.2.38. : <u>Homeomorphism</u>

Let X, Y be two topological spaces. A mapping $f : X \dashrightarrow Y$ is said to be a homeomorphism iff,

(i) f is bijective, that is, f is one-one and onto.

(ii) f and f^{-1} both are continuous.

1.2.39. : <u>Homeomorphic spaces</u>

A space X is said to be homeomorphic to another space Y if there exists a homeomorphism of X onto Y and then Y is said to be a homeomorph of X.

If X is homeomorphic to Y, we can write $X \approx Y$.

1.2.40. : Topological property

A property of a topological space X is said to be a topological property or topological invariant or intrinsic qualitative property if each homeomorph of X has that property whenever X has that property.

1.2.41. : Subspaces

Let A be a subset of a topological space X. The topology induced on A by the topology of X is that in which the open sets are the intersections with A of open sets of X. The set A with this topology is called a subspace of X.

The topology induced on A in known as Reletive topology.

Example :

Let $T = \{ \phi, \{1\}, \{3,4\}, \{1,3,4\}, \{2,3,4,5\}, X \}$ be a topology on $X = \{1,2,3,4,5\}$

Also, $A = \{1,4,5\} \subset X$ we then have,

$\phi \cap A = \phi$, $X \cap A = A$, $\{1\} \cap A = \{1\}$

$\{3,4\} \cap A = \{4\}$, $\{1,3,4\} \cap A = \{1,4\}$

$\{2,3,4,5\} \cap A = \{4,5\}$

Hence subspace of $A = \{ \phi, \{1\}, \{4\}, \{1,4\}, \{4,5\}, A \}$.

1.2.42. : <u>Quotient topology</u>

Let X be a topological space, R an equivalence relation on X, Y = X/R the quotient set of X with respect to the relation R and ϕ : X ---> Y is the cannonical mapping. The finest topology on Y for which ϕ is continuous is called the quotient of the topology on Y for which ϕ is continuous is called the qoutient of the topology of X by the relation R.

1.2.43. : <u>Quotient Spaces</u>

Let X be topological space, R an equivalence relation on R. The quotient space of X by R is the qoutient set X/R with the topology which is the quotient of the topology of X by the relation R.

1.2.44. : <u>Product topology</u>

Let X and Y be two topological spaces. The product topology X x Y is the topology having as basis, the collection B of all sets of the form u x v, where u is the basis of the topology X and v is the basis of the topology Y.

1.2.45. : Seperable spaces

A topological space X is said to be seperable if some countable subsets of X is dense in X.

1.2.46. : Kolmogoroff Spaces (T_0 - spaces)

A topological space X is said to be a Kolmogoroff space if it satisfies the following conditions :

Given any two distinct points x, y of X there is a neighbourhood of one of these points which does not contain the other.

A Kolmogoroff space is also known as T_0 - space.

Example : Every discrete space is a Kolmogoroff space.

1.2.47. : Frechet Spaces (T_1 - Spaces)

A topological space X is said to be Frechet space or T_1 -space iff given any pair of distinct points x and y of X there exists two open sets, one containing x but not y and the other containing y but not x, i.e., there exists open sets G and H such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$.

1.2.48. : <u>Hausdorff Spaces (T_2 - spaces)</u>

A topological space X is said to be a Hausdorff space or seperated space or T_2 -space iff for every pair of distinct points x, y of X, there exists disjoint neighbourhoods of x and y, that is, there exist neighbourhoods N of x and M of y such that $N \cap M = \phi$.

If (X, T) is a hausdorff space, then T is said to be a Hausdorff topology for X.

Example : If $T = \{ \phi , \{ a \}, \{ b \}, X \}$ be a topology on X where $X = \{ a, b \}$.

Here, $a \in \{ a \}$, $b \in \{ b \}$ and $\{a\} \cap \{b\} = \phi$

Hence (X, T) is a Hausdorff space.

1.2.49. : <u>Regular spaces</u>

A topological space X is said to be regular iff for every closed set $F \subset X$ and every point $p \in X$ with $p \in F$, there exists two open sets G and H such that

$F \subset G$, $p \in H$ and $G \cap H = \phi$

A regular T_1 -space is called T_3 -space.

1.2.50. : Normal spaces

A topological space X is said to be normal iff for every pair A, B of disjoint closed subsets of X there exists open sets G and H such that

$$A \subset G, \quad B \subset H \quad \text{and} \quad G \cap H = \phi$$

A normal T_1 -space is said to be a T_4 -space.

Remark :

T_4-spaces => T_3-spaces => T_2-spaces => T_1-space => T_0-spaces

1.2.51. : Completely Normal Spaces

A topological space X is said to be completely Normal iff for any two seperated subsets A and B of X, there exixts open sets G and H such that

$$A \subset G, \quad B \subset H \quad \text{and} \quad G \cap H = \phi$$

A completely normal space which is also T_1 is called a T_5 - space.

1.2.52. : Completely Regular Spaces

A topological space X is said to be Completely regular iff for every closed subset F of X and every point $x \in X - F$, there exists a continuous function $f : X \longrightarrow [0, 1]$ of R such that,

$$f(x) = 0 \quad \text{and} \quad f[F] = \{1\}$$

A completely regular T_1-space is known as Tychonoff space.

1.2.53. : Covers

Let X be a topological space and A be a subset of X. A collection $\{G_i\}$ is said to be cover of A if $A \subset \bigcup_i G_i$. If the class $\{G_i\}$ belongs to the open sets of the topological space X, then $\{G_i\}$ will be known as the open cover of A.

i.e. $\{G_i\}$ will be called the cover of the topological space X if $X = \bigcup_i G_i$

$\{G_i\}$ will be called the open cover of the topological space X if $\{G_i\} \in T$.

A covering $\{G_i\}$ is said to be simple if $\{G_i\}$ contains only a finite number of sets.

If G_1 is a covering of X and G_2 is another covering of X such that $G_2 \subset G_1$, then G_2 is a subcovering of G_1.

Example : If $G_1 = \{]-n, n[: n \in Z$ (set of integers) $\}$
Then G_1 is an open covering of the sets of real numbers R.

Also, $G_2 = \{]-n, n[: n \in Z^+$ (set of positive integers) $\}$ then $G_2 \subset G_1$, hence G_2 is the subcovering of G_1.

1.2.54. : Basic and Sub-basic open covers

An open cover of A is called as Basic (Sub-basic) open cover if all the open sets of the cover are contained in some given base (sub-base).

1.2.55. : Compact sets

A subset A of a topological space X is said to be compact if every open cover of A is reducible to a finite cover or in other words if every open cover of A contains a finite sub-cover.

1.2.56. : Compact Spaces

A topological space X is said to be compact iff every open cover of X has a finite sub-cover.

1.2.57. : Compact Sub-spaces

A subspace of a topological space, which is compact as a topological space in its own right, is said to be a compact sub-space.

1.2.58. : <u>Finite Intersection Property (FIP)</u>

A collection { G_α } of subsets of X is said to have FIP if the collection { G_α } is non-empty and each non-empty finite sub-collection of { G_α } has non-empty intersection.

<u>Result</u> :

A topological space X is compact iff every class { G_i } of closed subsets of X which satisfies the finite intersection property has, itself, a non-empty intersection.

1.2.59. : <u>Countably Compact Spaces</u>

A topological space X is said to be Countably Compact iff every countable open cover of X has a finite sub-cover.

1.2.60. : <u>Sequentially Compact Spaces</u>

A topological space X is said to be Sequentially compact iff every sequence in X has a convergent Sub-sequence.

1.2.61. : Lindelof Spaces

A topological space X is said to be a Lindelof space if every open covering of X contains a countable covering of X, that is, every space with a countable base is a Lindelof Space.

1.2.62. : Locally Compact Spaces

A topological space X is locally compact if for each point $x \in X$ and for each neighbourhood G of x, there is a compact neighbourhood V of x with $V \subset G$.

Remark :

Every compact space is locally compact, but the converse is not true, for example, every discrete space is locally compact, but not compact, if infinite.

1.2.63. : Compactification

Let (X, T) be a topological space and (X^*, T^*) be a compact topological space such that X is homeomorphic to a dense sub-space of X^*. Then (X^*, T^*) is called a compactification of (X, T).

1.2.64. : Refinement

Let { A_α | $\alpha \in J$ } and { B_β | $\beta \in M$ } be two coverings of a space X, where J and M are index sets. { A_α } is said to refine (or be a refinement of) { B_β } for each A_α, there is some B_β with $A_\alpha \subset B_\beta$ and { A_α } $\prec$ { B_β } where $\prec$ is a sign of refinement.

1.2.65. : Locally finite

A family A of sets in a space X is said to be locally finite iff every point of X has a neighbourhood which meets almost a finite number of members of A .

1.2.66. : Paracompact Spaces

A topological space X is said to be Paracompact iff every open cover of X has a locally finite open refinement.

Remark :

Every compact space is paracompact. Also, every discrete space X is paracompact, for the open covering formed by all sets consisting of a single point of X is locally finite and is finite than every open covering of X.

1.2.67. : Countably Paracompact

A space X is countably paracompact iff every countable open covering has a locally finite refinement.

A countably paracompact normal space ia called a binormal space.

1.2.68. : Connected Spaces

A topological space X is said to be Connected if it is not the union of two disjoint non-empty open sets.

If X is connected and if A, B are two non-empty open (respectively closed) subsets such that $A \cup B = X$, then $A \cap B \neq \phi$.

1.2.69. : Connected Sets

A subset A of a topological space X is said to be connected set if the subspace A of X is connected.

1.2.70. : Totally Disconnected Spaces

A topological space (X, T) is said to be totally disconnected iff for each pair $x, y \in X$, there exists a disconnection $A \cup B$ of X such that $x \in A$ and $y \in B$.

1.2.71. : <u>Extremally Disconnected Space</u>

A topological space X is called extremally disconnected if Cl(U) is open in X for every open set U of X, or, equivalently if every two disjoint open sets of X have disjoint closures.

1.2.72. : <u>Components</u>

The component of a point of a topological space X is the largest connected subset of X which contains this point. The components of a subset A of X are the components of the points of A, relative to the subspace A of X.

<u>Remarks</u> :

1. Every component of a topological space X is closed.
2. If a space is connected, the component of each point is the whole space.

If a space X is such that for each pair (x, y) of points of X there is a connected set containing x and y, then X is connected.

1.2.73. : <u>Locally Connected Spaces</u>

A topological space X is said to be locally connected if each point of X has a fundamental system of connected neighbourhoods.

1.2.74. : <u>Pathwise connected</u>

A space X is pathwise connected iff for any two points x and y in X, there exists a continuous function f : I ---> X such that f(0) = x and f(1) = y, such a function f is called a path from x to y.

1.2.75. : <u>Arcwise connected spaces</u>

A space X is arcwise connected iff for ant two points x and y in X, there is a homomorphism f : I ---> X such that f(0) = x and f(1) = y, the function f is called an arc from x to y.

1.2.76. : <u>Locally Pathwise Connected Spaces</u>

A space X is locally pathwise connected iff each point has a neighbourhood base consisting of pathwise connected sets.

1.2.77. : <u>Directed sets</u>

A set Λ is directed iff there is a relation $\geqslant$ on Λ satisfying

(i) $\lambda \geqslant \lambda$ for each $\lambda \in \Lambda$

(ii) if $\lambda_1 \geqslant \lambda_2$ and $\lambda_2 \geqslant \lambda_3$ then $\lambda_1 \geqslant \lambda_3$, $\forall \lambda_1, \lambda_2, \lambda_3 \in \Lambda$

(iii) If $\lambda_1, \lambda_2 \in \Lambda$ there is some $\lambda_3 \in \Lambda$ with $\lambda_1 \geqslant \lambda_3$, $\lambda_2 \geqslant \lambda_3$

The relation $\geqslant$ is known as the direction on Λ.

1.2.78. : <u>Nets</u>

A net in a set X is a function p : Λ ---> X where Λ is some directed set. The point p(λ) is denoted as x_λ.

A subnet of a net p : Λ ---> X is the composition p.φ where φ : M ---> Λ is an increasing cofinal function from the directed set M to Λ. That is

(i) $\varphi(\mu_1) \leqslant \varphi(\mu_2)$ where $\mu_1 \leqslant \mu_2$

(ii) for $\lambda \in \Lambda$ there is some $\mu \in$ M such that $\lambda \leqslant \varphi(\mu)$

1.2.79. : Ultranet

A net (x_n) in a set X is an ultranet iff for each sub-net E of X, (x_n) is either eventually in E or eventually in $X - E$.

1.2.80. : Filters

A filter on a set X is a set $\mathcal{F}$ of subsets of X which has the following properties :

F_I : Every subset of X which contains a set $\mathcal{F}$ belongs to $\mathcal{F}$. i.e., if $F \in \mathcal{F}$ and $F \subset G$ then $G \in \mathcal{F}$ where. $F, G \in X$.

F_{II} : Every finite intersection of sets of $\mathcal{F}$ belongs to $\mathcal{F}$. i.e., if $F, G \in \mathcal{F}$ then $F \cap G \in \mathcal{F}$.

F_{III} : The empty set ϕ is not in $\mathcal{F}$. i.e., $\phi \notin \mathcal{F}$

Example : If $X \neq \phi$, the set of subsets consisting of X alone is a filter on X. MOre generally, the set of all subsets of X which contain a given non-empty subset A of X is a filter on X.

1.2.81. : Neighbourhood Filter

In a topological space X, the set of all neighbourhoods of an arbitrary non-empty subset A of X is a filter, called the neighbourhood filter of A.

1.2.82. : Frechet Filter

If X is an infinite set, the complements of the finite subsets of X are the elements of a filter. The filter of complements of finite subsets of the set N of integers $\geqslant 0$ is called the Frechet filter.

1.2.83. : Comparison of Filters

Given two filters $\mathcal{F}$, $\mathcal{G}$ on the same set X, $\mathcal{G}$ is said to be finer than $\mathcal{F}$, or $\mathcal{F}$ is coarser than $\mathcal{G}$ if $\mathcal{F} \subset \mathcal{G}$. If also $\mathcal{F} \neq \mathcal{G}$, then $\mathcal{G}$ is said to be strictly finer than $\mathcal{F}$, or $\mathcal{F}$ is strictly coarser than $\mathcal{G}$.

Two filters are said to be comparable if one is finer than the other.

1.2.84. : Fixed and Free Filters

A filter $\mathcal{F}$ on X is fixed iff $\bigcap \mathcal{F} \neq \varphi$ and free iff $\bigcap \mathcal{F} = \varphi$.

1.2.85. : Base of a Filter

A set $\mathcal{B}$ of subsets of a set X ia said to be a base of the filter if it satisfies the following axioms :

(i) The intersection of two sets of $\mathcal{B}$ contains a set of $\mathcal{B}$. i.e., if $F \in \mathcal{B}$ and $H \in \mathcal{B}$, then there exists a $G \in \mathcal{B}$ such that $G \subset F \cap H$.

(ii) $\mathcal{B}$ is not empty and the empty subset of X is not in $\mathcal{B}$. i.e., $\mathcal{B} \neq \phi$ and $\phi \in \mathcal{B}$

Two filter bases are said to be equivalent if they generate the same filter.

Example : Let X = { a, b, c } Then { { a }, { a, b }, { a, c } } is a filter base in X.

1.2.86. : <u>Ultrafilters</u>

An ultrafilter on a set X is a filter $\mathcal{F}$ such that there is no filter on X which is strictly finer than $\mathcal{F}$.

1.2.87. ; <u>Ultrafilter Base</u>

A filter base on a set X is called an ultrafilter base iff it is a base of an ultrafilter.

1.2.88. : <u>Trace of a Filter</u>

Let $\mathcal{F}$ be a filter on a set X and A is a subset of X. Then the trace $\mathcal{F}_A$ of $\mathcal{F}$ on A is a filter iff each set of $\mathcal{F}$ meets A.

1.2.89. : Induced Filter

Let A be a subset of a set X and $\mathcal{F}$ a filter on X. If the trace of $\mathcal{F}$ on A is a filter on A, this filter is said to be induced by $\mathcal{F}$ on A.

1.2.90. : Limit of a Filter

Let X be a topological space and $\mathcal{F}$ a filter on X. A point $x \in X$ is said to be a limit point (or simply limit) of $\mathcal{F}$, if $\mathcal{F}$ is finer than the neighbourhood $\mathcal{B}(x)$ of x ; $\mathcal{F}$ is also said to converge (or to be convergent) to x. The point x is said to be a limit of a filter base $\mathcal{B}$ on X, and $\mathcal{B}$ is said to converge to x, if the filter whose base is $\mathcal{B}$ converges to x.

1.2.91. : Cluster Point of a Filter Base

In a topological space X, a point x is a clustre point of a filter base $\mathcal{B}$ on X if it lies in the closure of all the sets of $\mathcal{B}$.

1.3. EARLIER WORKS RELATED TO PRESENT INVESTIGATION

Noiri and Takashi [23] studied about S-closed space in a paper "A note on S-closed space". In their investigation, they have showed that, a space X is said to be S-closed if every cover of X by regular closed sets has a finite sun-cover. The main result is that an S-closed space in which every open set is the union of regular closed sets is extremally disconnected.

Woo, Moo Hajkwon, Taikyum [38] have introduced a paper "A note on S-closed spaces". Through this paper they showed that a topological space is said to be S-closed if every semi-open cover of X has a finite subfamily whose closure are a cover of the space. A space X is said to be quasi-H-closed if any open cover of the space has a finite subfamily whose closure cover the spaces.

Reilly and Vamanamurthy [29] have given a discussion "On semi-compact spaces". According to them a subset S of a topological space (X, T) is said to be semi-open if $S \subset Cl(Int(S))$. A topological space (X, T) is said to be semi-compact if each cover of X by semi open subsets has a finite sub-cover.

Zi. Qiu Yum [39] has discussed about necessary and sufficient condition for extreme disconnectedness for a locally S-closed space.

Ram Prasad [27] has been discussed on S-compact spaces. According to him, a subset B of a topological space X is said to be semi-open if there exists an open set U such that $U \subset B \subset Cl(U)$.

A topological space is said to be S-compact if each semi-open cover of X has a finite sub-cover.

In a paper entitled "Connectedness and strong semi-continuity", Reilly, Ivan and Vamanamurthy [28] stated that a subset S of a topological space (X, T) is called an α -set if $S \subset Int(Cl(Int(S)))$. It is known that (X, T^{α}) is a topological space, where T^{α} denotes the class of α -set on (X, T).

The concept of α -continuous and α -open mappings has been introduced by Mashhour, Hasanein and El-Deeb [19] . They showed that a subset of a topological space X is called an α -set if $S \subset (\overline{S^{o}})^{o}$ and for space X and Y , a mapping $f : X \longrightarrow Y$ is said to be α - continuous [α -open] if the inverge image of each open set in Y is an α -set.

Dasgupta and Lahiri [40] have investigated about the continuity of semi open and closed function with the result that if X and Y be first countable Hausdorff spaces and if Y is also compact and perfect, then every open and closed function from X to Y is continuous.

In a paper "On almost strongly θ -continuous function", Noiri, Kang and Sin Min [24] have introduced and investigated a new class of continuous functions, called almost strongly θ -continuous functions, which contains the class of strongly θ -continuous functions and is contained in the class of δ -continuous functions.

Noiri and Takashi [25] have introduced the idea of almost locally connected spaces. They showed that almost locally connectedness is preserved under almost open almost continuous surjections and that the union of regular open sets of an almost locally connected space is almost locally connected.

In the paper "Almost continuous functions", Bornor and Andrew [3] have shown that a function f : X ---> Y is almost continuous if for every $x \in X$ and for each open set $V \subset Y$ containing f(x), $Cl(f^{-1}(v))$ is a neighbourhood of x.

In the paper "Semi-continuous and semi-weakly continuous functions", Njastad [20] have shown that if $f : (X, T) \longrightarrow (Y, U)$ is a semi-weakly continuous iff $f : (X, T) \longrightarrow (Y, U)$ is semi-continuous, where U is the semi-regularization topology of Y.

Commaroto, Filippo, Lo Faro and Giovanni [4] have introduced a generalization of the concept of continuity in the paper " δ -continuous functions". Some inter relations between δ -continuous functions and other generalizations of continuity, such as almost continuity, θ -continuity etc. are obtained and some other properties of δ -continuity are studied in this paper.

In a paper "A note on strong compactness and s-closedness", Atia, R.H ; El-Deeb, S.N. ; Hasanein, I.A. [2] defined some properties of strong compactness and s-closedness. According to them, a subset S of a topological space X is pre-open if $S \subset Int(Cl(S))$ and semi-open if $S \subset Cl(Int(S))$. A space is said to be strongly compact if every pre-open cover of S has a finite sub-cover. A space is said to be s-closed if every semi-open cover has a finite subfamily whose closures form a cover. The authors Strengthen some results from two eaelier papers.

In the paper "On weakly closed functions", Smital, Jaroslav, Kubackova, Elena [34] defined a real valued real function f to be weakly closed at x_0 provided that whenever $\{x_i\}_{i=1}^{\infty}$ is a sequence in R that converges to x_0, the set of images $\{f(x_i)\}_{i=0}^{\infty}$ is closed. Let A_f denote the set of all points x at which f : R ---> R is not weakly closed and let $essA_f$ denote the set of all points $x \in A_f$ such that there does not exists a function g : R ---> R such that $f / R \setminus \{x\} = g / R \setminus \{x\}$ and g is weakly closed at x. Among the results obtained by the authors are the following :

Theorem - 1 : For any set $A \subset R$ there exists a f with $A = A_f$.

Theorem - 2 : For each f : R ---> R, $essA_f$ is an F_α -set.

Theorem - 3 : For any F_α -set A in R, there is a bounded f such that $essA_f = A$ and f is continuous outside the set A.

1.4. PRESENT DISCUSSION

In Chapter - II we have discussed some properties of s-closed and S-closed spaces. We have obtained some properties of extremally disconnected spaces and these are discussed in Chapter - III. In Chapter - IV, we have discussed some properties about strongly neighbourhood finite family of sets, almost continuous, δ-continuous and set-connected mappings.

CHAPTER - II

SOME CHARACTERIZATION OF s-CLOSED SPACES

2.1 : Introduction

The class of s-closed spaces has been introduced and studied by Di Maio and Noiri [16]. Gansten and Reilly [9] examine the relationship betwen the class of s-closed space and the more familier class of S-closed spaces which was introduced by Thompson [36] and studied by Cameron [5]. The purpose of this Capter is to obtain several characterization os s-closed and S-closed spaces.

2.2 Preliminaries

Throught the present note, by X we denote a topological space. Let A be a subset of X. The closure of A and the interior of A are denoted by Cl(A) and Int(A) respectively. A subset A is said to be Pre-open if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$. A subset A is called semi-open (respectively semi pre-open) if there exists an open (respectively pre-open) set U such that $U \subset A \subset \mathrm{Cl}(U)$. The family of all pre-open (respectively semi-open, semi pre-open) sets of X is denoted by PO(X) [respecticely

SO(X), SPO(X)]. Lavine[13] (respectively Andrijevic[1]) showed that a subset A of X is semi-open (respectively semi-preopen) iff $A \subset Cl(Int(A))$ [respectively $A \subset Cl(Int(Cl(A)))$].

The complement of a semi-open set is called semi-closed. The intersection of all semi-closed sets containing a subset A of X is called the semi-closure of A and is denoted by sCl(A). It is obvious that sCl(A) is semi-closed. The semi-interior of A, denoted by sInt(A), is difened by the union of all semi-open sets contained in A. A point x of X is said to be a θ-adherent point (respectively δ-adherent) point of A if $A \cap Cl(U) \neq \phi$ (respectively $A \cap Int(Cl(U) \neq \phi$) for every open set U containing x. The set of all θ-adherent (respectively δ-adherent) points of A is called the θ-closure (respectively δ-closure) of A and is denoted by $Cl_\theta(A)$ [respectively $Cl_\delta(A)$]

2.3 Regular open sets

Definition : 2.3.1 :

Let A be a subset of a topological space (X, T). A is said to be regular open (respectively regular closed) if $Int(Cl(A)) = A$ (respectively $Cl(Int(A)) = A$).

The family of all regular open sets of X is denoted by RO(X).

<u>Remark</u> : <u>2.3.1</u> : Every regular open set is open but the converse is not usually true.

Proof : Let $A \in RO(X)$

then $Int(cl(A)) = A$

$\Rightarrow$ $Int(Int(Cl(A))) = Int(A)$

$\Rightarrow$ $Int(Cl(A)) = Int(A)$ [$Int(Int(A)) = Int(A)$]

$\Rightarrow$ $A = Int(A)$

Hence A is open. But the converse is not usually true.

Proposition : 2.3.1 :

The intersection of two regular open sets is regular open.

Proof : Let A and B be two regular open sets of X. Then by definition,

$A = Int(Cl(A))$

and $B = Int(Cl(B))$

Now, $Int(Cl(A \cap B)) = Int(Cl(A) \cap Cl(B))$

$= Int(Cl(A)) \cap Int(Cl(B))$

$= A \cap B$

Hence the result.

Proposition : 2.3.2 :

If $\{ A_\alpha \}_{\alpha \in \nabla}$ be a collection of RO(X) then $\bigcup_{\alpha \in \nabla} A_\alpha \in RO(X)$.

Proof : Given that $\{ A_\alpha \}_{\alpha \in \nabla}$ be a collection of RO(X), i.e., A_α's are all RO(X).

Therefore, $Int(Cl(A_\alpha)) = A_\alpha,\ \alpha \in \nabla$

Now, $Int(Cl(\bigcup_{\alpha \in \nabla} A_\alpha)) = Int(Cl\{A_1 \cup A_2 \cup \text{-----}\}),\ \alpha = 1, 2, \text{--}$

$= Int(Cl(A_1) \cup Cl(A_2) \cup \text{-------})$

$= Int(Cl(A_1)) \cup Int(Cl(A_2)) \cup \text{-------}$

$= A_1 \cup A_2 \cup A_3 \cup \text{--------}$

$= \bigcup_{\alpha \in \nabla} A_\alpha$

Hence $\bigcup_{\alpha \in \nabla} A_\alpha \in RO(X)$.

Remark : 2.3.2

Every regular open set is semi-open but the converse is not usually true.

Proof : Let $A \in RO(X)$

then $A = Int(Cl(A))$

$\Rightarrow Int(Cl(A)) \subset A$

Now, $Int(Cl(A)) = Int(A)$

suppose, $U = Int(A)$ where U is an open set.

Then we have $U \subset A \subset Cl(U)$.

Therefore, A is semi-cpen. But the converse of this theorem is not necessarily true.

Proposition : 2.3.3

If $A \in RO(X)$, then $Cl(A) = Cl_\delta(A) = Cl_\theta(A)$

Proof : From the definition, it is obvious that

$Cl(A) \subset Cl_\delta(A) \subset Cl_\theta(A)$ for every subset A of X. Thus, to show the required result, we are to show that $Cl_\theta(A) \subset Cl(A)$.

Assume that $x \notin Cl(A)$, then $U \cap A = \phi$ for some open set U containing x.

We have $U \cap Cl(A) = \phi$ and have

$Cl(U) \cap Int(Cl(A)) = \phi$.

Since $A \in RO(X)$, we obtain $Cl(U) \cap A = \phi$.

This shows that $x \in X - Cl_\theta(A)$. Consequently, we have $Cl_\theta(A) \subset Cl(A)$ and hence we have

$Cl(A) = Cl_\delta(A) = Cl_\theta(A)$.

Proposition : 2.3.4

Let Y be a pre-open set of X and let U be a regular open set. Then $H = Y \cap U$ is regular open in the sub-space Y.

Proof : Since H is open in Y, then $H \subset Int_Y(Cl_Y(H))$. Now let there exists a point $x \in Int_Y(Cl_Y(H))$ such that $x \notin H$. Then there exists an open set V_Y in Y such that $x \in V_Y \subset Cl_Y(H) \subset Cl(H) \subset Cl(U)$, then V_Y is pre-open in X. Hence $x \in Int(Cl(U)) = U$. But $x \notin H$, $x \in Y$ and $H = Y \cap U \Rightarrow x \notin U$. This contradiction shows that $H = Int_Y(Cl_Y(H))$. Consequently H is regular open in the sub-space Y.

2.4 : Semi-regular sets

Definition : 2.4.1

A subset S of a space (X, T) is said to be semi-regular if it is both semi-open and semi-closed.

The family of all semi-regular sets of X is denoted by $SR(X)$. For each $x \in X$, the family of all semi-regular sets of X containing x is denoted by $SR(x)$.

Also, Di-Maio [15] defined a subset S of X to be semi-regular open if $S = sInt(sCl(S))$. On the other hand, Cameron [5] defined a subset S of a space X to be regular semi-open if there exists a regular open set U of X such that $U \subset S \subset Cl(U)$.

The following proposition shows that the above three notions are equivalent.

Proposition : 2.4.1

For a subset A of a space X, the following are equivalent :

(a) $A \in SR(X)$

(b) $A = sInt(sCl(A))$

(c) There exists a regular open set U of X such that $U \subset A \subset Cl(U)$.

Proof : (a) --> (b)

If $A \in SR(X)$, then we have $A = sCl(A) = sInt(A)$ Therefore, $sInt(sCl(A)) = sInt(A) = A$.

(b) --> (c)

Suppose that, $A = sInt(sCl(A))$. Since $Int(Cl(S)) \subset sCl(S)$ for every subset S of X, $Int(Cl(A)) \subset sInt(sCl(A)) = A$. Since $A \in SO(X)$, we have $A \subset Cl(Int(A))$. Thus we obtain

$$Int(Cl(A)) \subset A \subset Cl(Int(A)) \subset Cl(Int(Cl(A)))$$

Where, $Int(Cl(A))$ is regular open, since $Int(Cl(Int(Cl(A)))) = Int(Cl(A))$.

(c) --> (a)

It is obvious that $A \subset SO(X)$, then we have $Int(Cl(A)) = Int(Cl(U)) = U \subset A$ and hence A is semi-closed. Thus we have $A \in SR(X)$.

Proposition : 2.4.2

If $A \in SO(X)$, then $sCl(A) \in SR(X)$.

Proof : We know that, $sCl(A)$ is semi-closed. We are to show that $sCl(A) \in SO(X)$. Since $A \in SO(X)$, $U \subset A \subset Cl(U)$ for some open set U of X. Therefore, we have $U \subset sCl(U) \subset sCl(A) \subset Cl(U)$ and hence $sCl(A) \in SO(X)$. Hence $sCl(A) \in SR(X)$.

Definition : 2.4.2

A point $x \in X$ is said to be a semi-θ-adherent point of a subset S of X if $sCl(U) \cap S \neq \phi$ for every $U \in SO(x)$. The set of all semi-θ-adherent points of S is called semi-θ-closure of S and is denoted by $sCl_\theta(S)$. A subset S is called semi-θ-closed if $sCl_\theta(S) = S$.

Proposition : 2.4.3

Let A be a subset of a space X. Then we have

(a) If $A \in SO(X)$, then $sCl(A) = sCl_\theta(A)$.

(b) If $A \in SR(X)$, then A is semi-θ-closed.

Proof : (a) It is obvious that $sCl(A) \subset sCl_\theta(A)$. Assume that $x \notin sCl(A)$. Then for some $U \in SO(x)$, $A \cap U = \phi$ and hence $A \cap sCl(U) = \phi$, since $A \in SO(X)$. This shows that $x \notin sCl_\theta(A)$. Therefore, $sCl(A) = sCl_\theta(A)$.

(b) $A \in SR(X)$, which gives A is semi-open and semi-closed. If $A \in SO(X)$, we have $sCl(A) = sCl_\theta(A)$. Also if A is semi-closed then $sCl(A) = A$.

Hence $sCl_\theta(A) = sCl(A) = A$ and consequently A is semi-θ-closed.

Proposition : 2.4.4

If O is open in a space X, then

$$sCl(O) = Int(Cl(O)).$$

Proof : For every subset S of X, $Int(Cl(S)) \subset sCl(S)$, (Noiri [21]). We show the opposite inclusion. Assume that $x \notin Int(Cl(O))$. Then $x \in Cl(Int(X-O)) \in SO(X)$. Since O is open, we have $O \subset Int(Cl(O))$ and $O \cap Cl(Int(X-O)) = \phi$. This shows that $x \notin sCl(O)$. Therefore, we obtain

$$sCl(O) = Int(Cl(O)).$$

2.5 : s-closed and S-closed spaces

Definition : 2.5.1

A space (X, T) is said to be s-closed if for every cover $\{V_\alpha : \alpha \in \nabla\}$ of X by semi-open sets of X, there exists a finite subset ∇_0 of ∇ such that

$$X = \bigcup \{ sCl(V_\alpha) : \alpha \in \nabla_0 \}.$$

In definition 2.5.1 the space (X, T) is called S-closed if $sCl(V_\alpha)$ is replaced by $Cl(V_\alpha)$. It is obvious that every s-closed space is S-closed and in Maio and Noiri [1] we have, if X is extremally disconnected then $Cl_X(S) = sCl_X(S)$ for every semi-open set S in X. Among weakly T_2-spaces the class of S-closed spaces coincides with the class of s-closed spaces.

A filter base $\mathcal{F}$ on X is said to be SR-converse to $x \in X$ if for each $V \in SR(x)$ there exists $F \in \mathcal{F}$ such that $F \subset V$. A filter base $\mathcal{F}$ is said to SR-accumulate at $x \in X$ if $V \cap F \neq \phi$ for every $V \in SR(x)$ and $F \in \mathcal{F}$.

Proposition : 2.5.1

For a space (X, T) the following are equivalent :

(a) X is s-closed.

(b) Every cover of X by semi-regular sets has a finite subcover.

(c) Every maximal filter base SR-converse to some point of X.

(d) Every filter base SR-accumulates at some point of X.

(e) For every family $\{ V_\alpha : \alpha \in \nabla \}$ of semi-regular sets such that $\bigcap \{ V_\alpha : \alpha \in \nabla \} = \phi$, there exists a finite subset ∇_0 of ∇ such that $\bigcap \{ V_\alpha : \alpha \in \nabla_0 \} = \phi$.

Proof : (a) --> (b)

From the definition of s-closed space, it is obvious.

(b) --> (c)

Suppose $\mathcal{F}$ be a maximal filter base of X. Assume that $\mathcal{F}$ does not SR-converse to any point of X. Then $\mathcal{F}$ does not SR-accumulate at any point of X. For each $x \in X$, there exists $F_x \in \mathcal{F}$ and $V_x \in SR(x)$ such that $V_x \cap F_x = \phi$. The family $\{ V_x : x \in X \}$ is a cover of X by semi-regular sets of X. By (b), there exists a finite number of points $x_1, x_2, \text{---------}, x_n$ such that $X = \bigcup \{ V_{x_i} : i = 1,2,3,\text{------},n \}$. Since $\mathcal{F}$ is a filter base on X, there exists $F_0 \in \mathcal{F}$ such that

$F_0 \subset \cap \{ F_{x_i} : i = 1,2,3,\text{--------},n \}$. Therefore, we have $F_0 = \phi$. This is a contradiction. Hence the result.

(c) --> (d)

Let $\mathcal{F}$ be a filter base on X and $\mathcal{F}_0$ be a maximal filter base such that $\mathcal{F} \subset \mathcal{F}_0$. By (c), $\mathcal{F}_0$ SR-converges to some $x \in X$. For every $F \in \mathcal{F}$ and every $V \in SR(x)$, there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subset V$. Therefore, we obtain $V \cap F \supset F_0 \cap F \neq \varphi$. This shows that $\mathcal{F}$, SR-accumulates at x.

(d) --> (e)

Let $\{ V_\alpha : \alpha \in \nabla \}$ be a family of semi-regular sets such that $\cap \{ V_\alpha : \alpha \in \nabla \} = \varphi$. Let $\Gamma(\nabla)$ denotes the family of all finite subsets of ∇. Assume that $\cap \{ \nabla_\alpha : \alpha \in \gamma \} \neq \varphi$ for every $\gamma \in \Gamma(\nabla)$. Then the family $\mathcal{F} = \left\{ \bigcap_{\alpha \in \gamma} V_\alpha \mid \gamma \in \Gamma(\nabla) \right\}$

is a filter base on X. By (d), F SR-accumulates at some $x \in X$. Since $\{ X - V_\alpha : \alpha \in \nabla \}$ is a cover of X, $x \in X - V_{\alpha_0}$ for söme $\alpha_0 \in \nabla$. Therefore, we have $X - V_{\alpha_0} \in SR(x)$ and $V_{\alpha_0} \in \mathcal{F}$. This is a contradiction.

(e) --> (a)

Let $\{ V_\alpha : \alpha \in \nabla \}$ be a cover of X by semi-open sets of X, by proposition 2.4.2, $\{ sCl(V_\alpha) \mid \alpha \in \nabla \}$ is a semi-regular cover of X. Thus $\{ X - sCl(V_\alpha) \mid \alpha \in \nabla \}$ is a family of semi regular sets of X having the empty intersection. By (e), there exists a finite subset ∇_0 of ∇ such that,

$$\bigcap \{ X - sCl(V_\alpha) \mid \alpha \in \nabla_0 \} = \phi .$$

Hence $X = \bigcup \{ sCl(V_\alpha) \mid \alpha \in \nabla_0 \}$. which shows that X is s-closed.

Proposition : 2.5.2

Every infinite topological space X can be represented as a closed subspaces of a space X which is s-closed.

Proof : Let Z be an infinite T_1 - space and let $Z_1 = Z \times \{1\}$ and $Z_2 = Z \times \{2\}$. We may assume that $Y \cap (Z_1 \cup Z_2)$ is empty. Let $X = Y \cup Z_1 \cup Z_2$. for $i = 1, 2$ let $\mathcal{B}_i = \{ W_i \subset Z_i \mid W_i = V \times \{i\}$ for some open subset V of $Z \}$. Let $\mathcal{B}_3 = \{ G \subset X \mid G = U \cup e_1 \cup e_2$ where $U \subset Y$ is opened in Y and e_1 respectively e_2 are co-finite subsets of Z_1 respectively Z_2 $\}$. We know that a topology on X is defined by taking $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ as a base. Clearly Z_1 and Z_2 are open in X and Y is a closed subspaces of X .

For $i = 1, 2$ we obviously have $Cl_X\ Z_i = Y \cup Z_i$ and $sCl_X\ Z_i = Z_i$ for each $y \in Y$, if $S_Y = Z_1 \cup \{ y \}$ then S_Y is semi-open in X and $sCl_X\ S_Y = S_Y$.

Hence $\{ S_Y \mid y \in Y \} \cup \{ Z_2 \}$ is a semi-open cover of X which has no finite sub-family the semi-closures of whose members cover X. Thus X is not s-closed.

Proposition : 2.5.3

There exists S-closed which are not s-closed.

Proof : Let Z be an infinite S-closed T_1-space. Since $Z_1 \cup Z_2$ is dense in X, it follows that X is closed (Noiri [21]). In particular, if Z is an infinite set covering the co-finite topology then X is even a connected S-closed T_1 -space which fails to be s-closed.

Defination : 2.5.2

A space is said to be semi-T_2 if every pair of distinct points can be seperated by disjoint semi-open sets.

Corollary : 2.5.1

There exists semi-T_2 S-closed spaces which are not s-closed.

Proof : If Y is an infinite discrete space and Z is an infinite hausdorff S-closed space then X is T_1 and S-closed but not s-closed. [from proposition 2.5.2 and proposition 2.5.3] . It is easily checked that X is semi- T_2 .

For example, if y_1 , $y_2 \in Y$ and $y_1 \neq y_2$ then $Z_1 \cup \{ Y_1 \}$ and $Z_2 \cup \{ Y_2 \}$ are disjoint semi-open sets containing y_1 and y_2 respectively. Similarly we can show the other cases also.

Definition : 2.5.3

A space (X, T) is said to be s-regular (respectively semi-regular [8]) if for each closed (respectively semi-closed) set F and each point $x \notin F$, there exists disjoint semi-open sets U and V such that $x \in U$ and $F \subset V$. A space (X, T) is said to be s-compact [6] if every semi-open cover of X has a finite subcover. It follows from Carnahan [6] that s-compactness implies s-closedness but not conversely and s-closedness neither implies compactness nor is implied by compactness.

Proposition : 2.5.4

If (X, T) is a semi-regular (respectively s-regular) s-closed space, then it is s-compact (respectively compact).

Proof : Let $\{ V_\alpha : \alpha \in \nabla \}$ be a semi-open (respectively open) cover of X. For each $x \in X$, there exists $\alpha(x) \in \nabla$ such that $x \in V_{\alpha(x)}$. Since X is semi-regular (respectively s-regular), there exists $W_x \in SO(x)$ such that $sCl(W_x) \subset V_{\alpha(x)}$. The family $\{ W_x : x \in X \}$ is a semi open cover of X. By s-closedness of X, there exists a finite number of points $x_1, x_2, x_3, \text{-------}, x_n$ of X such that

$$X = \bigcup_{i=1}^{n} sCl(W_{x_i}) \subset \bigcup_{i=1}^{n} V_\alpha(x_i)$$

Therefore, X is s-compact (respectively compact).

6 : Sets, s-closed Relative to a space

Definition : 2.6.1

A subset S of a space X is said to be s-closed relative to X if for every cover $\{ V_\alpha : \alpha \in \nabla \}$ of S by semi-open sets of X, there exists a finite subset ∇_0 of ∇ such that $S \subset \bigcup \{ sCl(V_\alpha) : \alpha \in \nabla_0 \}$

Proposition : 2.6.1

If K is a semi θ-closed sets of an s-closed space (X, T), then K is s-closed relative to X.

Proof : Let $\{ V_\alpha : \alpha \in \nabla \}$ be a cover of K by semi-regular sets of X. For each $x \in X - K$, there exists $U(x) \in SO(x)$ such that $sCl(U(x)) \subset X - K$. By proposition 2.4.2, $sCl(U(x)) \in SR(x)$ for each $x \in X$ and the family $\{ sCl(U(x)) : x \in X - K \} \cup \{ V_\alpha : \alpha \in \nabla \}$ is a cover of X by semi-regular sets of X. Since X is s-closed, there exists a finite subset ∇_0 of ∇ such that

$$K \subset \cup \{ V_\alpha \mid \alpha \in \nabla_0 \}$$

Thus K is s-closed relative to X.

Definition : 2.6.2

A space (X, T) is said to be weakly-hausdorff (simply weakly T_2)[35] if every point of X is the intersection of regular closed sets of X.

Proposition : 2.6.2

Let (X, T) be a weakly-T_2 space. If K is s-closed relative to X. then K is semi-θ closed.

Proof : Let $x \in X - K$. for each $x \in K$, there exists a regular closed set F_K such that $K \in F_K$ and $x \notin F_K$. Since $F_K \in SR(x)$ and $K \subset \cup \{ F_k : k \in K \}$, by proposition 4.1 of Maio and Noiri [16] we have $K \subset \cup \{ F_K : k \in K_0 \}$ for some finite subset K_0

of K. Now putting $V = \cap \{ X - F_k \mid k \in K_0 \}$. Then $V \in SR(x)$ and $V \cap K = \emptyset$. Therefore, $x \notin sCl_\theta(K)$ and hence K is semi-θ closed.

7: The Quasi-irresolute Images of s-closed spaces

Definition : 2.7.1

A function $f : X \to Y$ is said to be quasi-irresolute if for each $x \in X$ and $V \in SO(f(x))$ there exists $U \in SO(x)$ such that $f(U) \subset sCl(U)$.

A function $f : (X, T) \to (Y, \sigma)$ is said to be irresolute (respectively semi-contineous) if $f^{-1}(V) \in SO(X)$ for every $V \in SO(Y)$ (respectively $V \in \sigma$).

Proposition : 2.7.1

If $f : X \to Y$ is a quasi-irresolute function and K is s-closed relative to X, then f(K) is closed relative to Y.

Proof : Let $\{ V_\alpha \mid \alpha \in \nabla \}$ be a cover of f(K) by semi-regular sets of Y. Since f is quasi-irresolute, $\{ f^{-1}(V_\alpha) \mid \alpha \in \nabla \}$ is a cover of K by semi-regular sets of X. (Maio and Noiri [16]). By proposition 2.5.1, there exists a finite subset ∇_0 of ∇ such that

$$K \subset \cup \{ f^{-1}(V_\alpha) \mid \alpha \in \nabla_0 \}$$

Therefore, $f(K) \subset \bigcup\{ V_\alpha : \alpha \in \nabla_0\}$. It follows from proposition 2.5.1 that f(K) is s-closed relative to Y.

A function f : X ---> Y is said to be semi-θ closed if f(F) is semi-θ-closed in Y for every semi-θ-closed set F of X.

Proposition : 2.7.2

If f : X ---> Y is a quasi-irresolute function, X is s-closed and Y is weakly - T_2, then f is semi-θ closed.

Proof : Let F be a semi-θ-closed set of X. By proposition 2.6.1, F is s-closed relative to X and f(F) is s-closed relative to Y (Maio and Noiri [16]). Since Y is weakly- T_2, by prpposition 2.6.2., f(F) is semi-θ-closed and hence f is semi-θ-closed.

C H A P T E R - III

Extremally Disconnected Spaces

3.1. : Introduction

Levine [13] have defined semi-open sets and utilizing this semi-open sets, Sivaraj [32] has obtained some characterisation of extremally disconnected spaces. Pre-open set and semi pre-open sets are defined by Mashhour et al [18] and Andrijevic [1]. Now, using pre-open sets and semi-pre-open sets, several characterisation of extremally disconnected spaces are obtained in this chapter.

3.2 : Preliminaries

A subset A of a topological space X is called semi-open, if there exists an open set U of X such that $U \subset A \subset Cl(U)$, where $Cl(U)$ denotes the closure of U in X. The collection of all semi-open sets in a topological space X is denoted by SO(X). A subset A is said to be pre-open if $A \subset Int(Cl(A))$. A subset A is called semi-pre-open if there exists an pre-open set U of X such that $U \subset A \subset Cl(U)$. Abd. El. Mansef et al [18] have introduced

β-open sets and β-open are equivalent to that of semi pre-open sets. The family of all pre-open (respectively semi-pre-open) sets of X is denoted by PO(X) (respectively SPO(X)).

The complement of semi-open set is called semi closed [7]. The intersection of all semi-closed sets containing a subset A of X is called the semi-closure [7] of A and is denoted by sCl(A). A point x of X is said to be a θ-adherent [37] (respectively δ-adherent) point of A if $A \cap Cl(U) \neq \phi$ (respectively $A \cap Int(Cl(U)) \neq \phi$) for every open set U containing x. The set of all θ-adherent (respectively δ-adherent) points of A is denoted by $Cl_{\theta}(A)$ (respectively $Cl_{\delta}(A)$).

Lemma : 3.3.1 :

For any open set A in a space X,

$sCl(A) = Int(Cl(A))$

Proof : For every subset S of X, $Int(Cl(S)) \subset sCl(S)$. Assume that $x \notin Int(Cl(A))$. Then $x \in Cl(Int(X - A)) \in SO(X)$. Since A is open, we have, $A \subset Int(Cl(A)$ and $A \cap Cl(Int(X - A)) = \phi$. This shows that $x \notin sCl(A)$. Therefore, we hwve, $sCl(A) = Int(Cl(A))$.

Lemma : 3.3.2

If $A \in PO(X)$, then $Cl(A) = Cl_{\delta}(A) = Cl_{\theta}(A)$.

<u>Proof</u> : It is obvious that $Cl(S) \subset Cl_{\delta}(S) \subset Cl_{\theta}(S)$ for every subset S of X. Thus, it remains to show that $Cl_{\theta}(A) \subset Cl(A)$. Assume that $x \notin Cl(A)$, then $U \cap A = \phi$ for some open set U containing x. We have $U \cap Cl(A) = \phi$ and hence $Cl(U) \cap Int(Cl(A)) = \phi$. Since $A \in PO(X)$ we obtain $Cl(U) \cap A = \phi$. This shows that $x \in X - Cl_{\theta}(A)$ $\Rightarrow x \notin Cl_{\theta}(A) \Rightarrow Cl_{\theta}(A) \subset Cl(A)$. Consequently we have,

$$Cl(A) = Cl_{\delta}(A) = Cl_{\theta}(A).$$

Lemma : 3.3.3

If $A \in SPO(X)$ then $Cl(A) = Cl_{\delta}(A)$.

<u>Proof</u> : It is obvious that $Cl(A) \subset Cl_{\delta}(A)$. Assume that $x \in X - Cl(A)$, then $U \cap A = \phi$ for some open set U containing x. Since U is open, $Cl(U) \cap Int(Cl(A)) = \phi$ and hence we have $Int(Cl(U)) \cap Cl(Int(Cl(A))) = \phi$. Again since $A \in SPO(X)$, we obtain $Int(Cl(U)) \cap A = \phi$. This shows that $x \in X - Cl_{\delta}(A)$. Therefore we have $Cl_{\delta}(A) \subset Cl(A)$ and hence $Cl(A) = Cl_{\delta}(A)$.

3.3 : Extremally Disconnected Spaces :

Definition : 3.3.1

A topological space X is called extremally disconnected (briefly E.D.) if Cl(U) is open in X for every open set U of X.

Proposition : 3.3.1

The following are equivalent for a space X.

(a) X is E.D.

(b) For each closed set F in X Int(F) is closed.

(c) For each disjoint pair of open sets U, V in X, we have $Cl(U) \cap Cl(V) = \phi$.

Proof : (a) --> (b)

Let F be a closed set in X then X - F is open in X. Since X is E.D., Cl(X - F) is open in X and hence X - Int(F) (In X, Cl (X - A) = X - Int(A)) is also open and consequently Int(F) is closed in X.

(b) --> (c)

Let U and V be two open sets of X such that $U \cap V = \phi$, then X - U and X - V are closed in X. Now $(X - U) \cup (X - V) = X - (U \cap V)$ is closed.

$= U \cap V$ is open.

Therefore, $Cl(U \cap V) = Cl(\phi) = \phi$.

$\Rightarrow Cl(U) \cap Cl(V) = \phi$.

(c) --> (a)

$Cl(U) \cap Cl(V) = \phi$ for any two open sets U and V in X.

We know that ϕ is open set, therefore Cl(U) and Cl(V) must be open in X. Hence X is E. D.

Proposition : 3.3.2

A space (X, T) is extremally disconnected iff $Cl(A) = sCl(A)$ for every $A \in SO(X)$.

Proof : **Necessity :**

It is obvious that for every subset S of X, $Int(Cl(S)) \subseteq sCl(S) \subset Cl(S)$. Since X is E.D., Cl(A) is open in X for every $A \in PO(X)$. Therefore we have $sCl(A) = Cl(A)$ for every $A \in PO(X)$.

Sufficiency : For every $A \in T$, $A \in SO(X)$ and by Lemma : 3.3.1, $Cl(A) = sCl(A) = Int(Cl(A))$. This shows that Cl(A) is open for every $A \in T$. Hence X is E.D.

Proposition : 3.3.3

The following are equivalent for a space X :

(a) X is E.D.

(b) The closure of every semi-pre-open set of X is open.

(c) The δ-closure of every semi-pre-open set of X is open.

(d) The δ-closure of every pre-open set of X is open.

(e) The θ-closure of every pre-open set of X is open.

(f) The closure of every pre-open-set of X is open.

Proof : (a) --> (b)

Let $A \in SPO(X)$ which gives A is open in X. Since X is E.D., $Cl(A)$ is open in X.

(b) --> (c)

Since $Cl(A)$ is open X for $A \in SPO(X)$ and using Lemma-3.3.3, we have $Cl_{\delta}(A)$ is open in X.

(c) --> (d)

Since every pre-open set of X is semi-pre-open, therefore as $Cl_{\delta}(A)$ is open for $A \in SPO(X)$, $Cl_{\delta}(A)$ is also open for $A \in PO(X)$.

(d) --> (e)

For $A \in PO(X)$ and using Lemma-3.3.2 we have $Cl_{\theta}(A)$ is open.

(e) --> (f)

Using Lemma-3.3.2, it is obvious.

(f) --> (a)

For every $A \in SO(X)$, $Cl(A)$ is open. Since $A \in SO(X) \Rightarrow A$ is open and consequently $Cl(A)$ is open. Hence X is E.D.

Proposition : 3.3.4

For every $A \in PO(X) \cup SO(X)$ and $sCl(A) = Cl_{\theta}(A)$ then X is E.D.

Proof : First, let A be any pre-open set of X. By Lemma-3.3.2 we have $Int(Cl(A)) = sCl(A) = Cl_{\theta}(A) = Cl(A)$. Therefore, $Cl(A)$ is open in X and hence it follows by proposition-3.3.3 that, X is E.D.

Next, let A be any semi-pre-open set of X. We have $sCl(A) \subset Cl(A) \subset Cl_{\theta}(A) = sCL(A)$ and hence $sCl(A) = Cl(A)$. Therefore, it follows from proposition-3.3.2 that X is E.D.

Proposition : 3.3.5

The following are equivalent for a space X :

(a) X is E.D.

(b) $sCl(A) = Cl(A)$ for every $A \in SPO(X)$.

(c) $sCl(A) = Cl_{\delta}(A)$ for every $A \in SPO(X)$.

(a) --> (b)

For every subset S of X, $Int(Cl(S)) \subset sCl(S) \subset Cl(S)$. Since X is E.D., by Proposition-3.3.3, $Cl(A)$ is open in X for every $A \in SPO(X)$. Therefore, we have $sCl(A) = Cl(A)$ for every $A \in SPO(X)$. Therefore, we have $sCl(A) = Cl(A)$ for every $A \in SPO(X)$.

(b) --> (c)

From Lemma-3.3.2 we have $Cl(A) = Cl_\delta(A)$ for $A \in SPO(X)$. Hence $sCl(A) = Cl(A) = Cl_\delta(A)$.

$\Rightarrow$ $sCl(A) = Cl_\delta(A)$.

(c) --> (a)

Let U and V be any two disjoint open sets, then we have $sCl(U) \cap V = \phi$. Since $sCl(U) \in SO(X)$, we have $sCl(U) \cap sCl(V) = \phi$. By Lemma-3.3.2, we obtain $Cl(U) \cap Cl(V) = \phi$. This shows that X is E.D.

Proposition : 3.3.6

The following are equivalent for a space X :

(a) X is E.D.

(b) If $A \in SPO(X)$, $B \in SO(X)$ and $A \cap B = \phi$, then $Cl(A) \cap Cl(B) = \phi$.

(c) If $A \in SPO(X)$, $B \in SO(X)$ then $A \cap B = \phi$ and $Cl_\delta(A) \cap Cl_\delta(B) = \phi$.

(e) --> (a)

Since every open set is pre-open and semi-open, therefore, A and B both are open. Also it is given that $A \cap B = \phi$ and $Cl(A) \cap Cl(B) = \phi$. Hence X is E.D.

Proposition : 3.3.7

The following are equivalent for a space X :

(a) X is E.D.

(b) If $A \in SO(X)$ and $B \in SPO(X)$ then
$Cl(A) \cap Cl(B) = Cl(A \cap B)$.

(c) If $A \in SO(X)$ and $B \in SPO(X)$, then
$A \cap B \in SPO(X)$.

Proof : **(a) --> (b)**

Let $A \in SO(X)$ and $B \in SPO(X)$. By Proposition -3.3.3, Cl(B) is open in X and we obtain

$$Cl(A) \cap Cl(B) = Cl(Int(A)) \cap Cl(B).$$

We know that if U is open in X then $U \cap Cl(S) \subset Cl(U \cap S)$ for every subset S of X. Hence,

$$Cl(A) \cap Cl(B) = Cl(Int(A)) \cap Cl(B)$$
$$\subset Cl(Int(A) \cap Cl(B)) \subset Cl(A \cap B).$$

Therefore, we have $Cl(A) \cap Cl(B) = Cl(A \cap B)$.

(b) --> (c)

Let $A \in SO(X)$ and $B \in SPO(X)$. Then we have

$A \cap B \subset Cl(Int(A)) \cap Cl(Int(Cl(B)))$

$= \quad Cl(Int(A)) \cap Int(Cl(B))$

$= \quad Cl(Int(A \cap Cl(B)))$

$\subset \quad Cl(Int(Cl(A) \cap Cl(B)))$

$= \quad Cl(Int(Cl(A \cap B)))$.

This shows that $A \cap B \in SPO(X)$.

(c) --> (a)

It is sufficiet to show that $Cl(A) \cap Cl(B) = Cl(A \cap B)$ for all open set A and B. Let A and B be any two open sets of X then Cl(A) and Cl(B) are semi-open and hence $Cl(A) \cap Cl(B) \in SPO(X)$. Therefore, we have,

$Cl(A) \cap Cl(B) \subset Cl(Int(Cl(A) \cap Cl(B)))$

$= \quad Cl(Int(Cl(A)) \cap Int(Cl(B)))$

$\subset \quad Cl(Cl(A) \cap Int(Cl(B)))$

$\subseteq \quad Cl(A \cap Int(Cl(B)))$

$\subset \quad Cl(A \cap Cl(B))$

$\subset \quad Cl(A \cap B)$.

Consequently we have,

$Cl(A) \cap Cl(B) \quad = \quad Cl(A \cap B)$.

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CHAPTER - IV

ON STRONGLY Nbd-FINITE FAMILY OF SETS, ALMOST CONTINUOUS, δ-CONTINUOUS AND SET CONNECTED MAPPINGS

4.1 : Introduction:

In this chapter, we study some properties of strongly nbd. finite family of sets, almost continuous, δ - continuous and set connected mappings. Almost continuous mapping was first defined by Husain [10] and also by Singal and Singal [31]. Long and Carnahan [14] also characterised some properties about almost continuous mapping. δ-continuous mapping was first defined by Noiri [26] and set connected mapping was first defined by Kwak [11].

4.2 : Preliminaries :

In this cahpter, X and Y always denote topological spaces. Let S be a subset of X. The closure (respectively interior) of S will be denoted by Cl(S) (resrectively Int(S)). A subset S of X is called semi-open (respectively regular open, pre-open) if $S \subset Cl(Int(S))$ (respectively $S = Int(Cl(S))$, $S \subset Int(Cl(S))$).

The complement of semi-open (respectively regular open, pre-open) set is called semi-closed (respectively regular closed, pre-closed). The family of all semi-open (respectively regular open, pre-open) sets will be denoted by SO(X) (respectively RO(X), PO(X)). It is clear from Chapter - II that, regular openness implies openness implies pre-openness, and semi-openness but the converses are not true.

4.3 : Strongly nbd-finite family

Definition : 4.3.1

A family $\{ A_m : m \in M \}$ of subset of X is said to be strongly nbd-finite if for each x in X, there is an open set V containing x, satisfying one of the following conditions :

(a) $V \cap A_m = \phi$ for every m in M.

(b) There is a non-empty finite subset N of M such that,

(i) $V \cap A_m \neq \phi$ for every m in M.

(ii) $V \cap A_m \subset A_k$ for every m, k with m in M and k in N and

(iii) $V \cap A_m = \phi$ for every $m \notin N$.

Every strongly nbd-finite family is nbd-finite. however, a nbd-finite family need not be strongly nbd-finite as shown by the following example :

Let $X = \{ a, b, c \}$

and $T = \{ \phi , \{ a \}, \{ b, c \}, X \}$

Then $\{ \{ b \}, \{ c \} \}$ is nbd-finite but not strongly nbd-finite.

Proposition : 4.3.1

If $\{ A_m ; m \in M \}$ is strongly nbd-finite, then $\{ sCl(A_m) ; m \in M \}$ is also strongly nbd-finite.

Proof : Let $x \in X$ and V be an open set containing x, then from definition - 4.3.1(a), $V \cap sCl(A_m) = \phi$ for every m in M. So, let us assume that Definition-4.3.1(b) holds, then it is enough to prove that $V \cap sCl(A_m) \subset sCl(A_k)$ where $m \in M$ and $k \in N$.

Let $y \in V \cap sCl(A_m)$, $m \in M$. Let S be any semi-open set containing y. Then $V \cap S$ is semi-open (Theorem 1.9 of Crossely and Hildebrand [7]). Hence $V \cap S \cap A_m \neq \phi$. If $k \in N$, then by definition-4.3.1(b) (ii), $V \cap S \cap A_k \neq \phi$ which implies $S \cap A_k \neq \phi$ and hence $y \in sCl(A_k)$. This completes the proof.

Proposition : 4.3.2

If $\{ A_m ; m \in M \}$ is a strongly nbd-finite family of semi-closed sets in X, then $B = \bigcup_{m \in M} A_m$ is semi-closed.

Proof : Let $x \in X - B$ and V be an open set containing x. if Definition-4.3.1(a) holds, taking $S = V$ and if Definition-4.3.1(b) holds taking $S = \bigcup_{m \in N} [V \cap (X - A_m)]$. Since union of semi-open sets is semi-open, S is semi-open. Clearly, $x \in S$ and hence $B \cap S = \phi$ so that $x \in S \subset X - B$ which implies that $x - B$ is semi-open and hence B is semi-closed.

Definition : 4.3.2

A subset S of a topological space X is said to be an α-set (Alpha set) if $S \subset Int(Cl(Int(S)))$.

Proposition : 4.3.3

Let $\{ A_m ; m \in M \}$ be an α-set covering of X and B, a subset of X. Then B is semi-open (respectively semi-closed) in X iff $B \cap A_m$ is semi-open (respectively semi-closed) in the subspace A_m for every m.

Proof : Necessary part :

Let B be an semi-open set in X. Then $B \cap A_m$ is semi-open in X. (By Proposition-1 of Njastad[20]) and $B \cap A_m$ is semi-open in A_m (Theorem-1 of Noiri[21]), $B \cap A_m$ is semi-open in A_m.

Now let B be semi-closed in X. Then X - B is semi-open in X. Again, by Proposition-1 of Njastad[20], $(X - B) \cap A_m$ is semi-open in X and hence semi-open in A_m. As $(X - B) \cap A_m = X - (B \cap A_m)$, $B \cap A_m$ is semi-closed in A_m.

Sufficient part :

Assume that $B \cap A_m$ is semi-open in A_m for every m. Then $B \cap A_m$ is semi-open in X for every m. Since $B = B \cap X = B \cap (\bigcup_{m \in M} A_m) = \bigcup_{m \in M} (B \cap A_m)$, B is semi-open in X. Assume that $B \cap A_m$ is semi-closed in A_m for each m. Then $A_m - (B \cap A_m)$ is semi-open in A_m and hence semi-open in X. Hence,

$$X - B = \bigcup_{m \in M} (A_m - (B \cap A_m))$$

Which implies that B is semi-closed in X. This completes the proof.

Proposition : 4.3.4

Let $\{A_m : m \in M\}$ be an covering of X such that all A_m are semi-closed and form a strongly nbd-finite family. Let B be a subset of X. If $B \cap A_m$ is semi-closed (respectively semi-open) in A_m for every m, then B is semi-closed (respectively semi-open) in X.

Proof : If $B \cap A_m$ is semi-closed in A_m for every m, then $B \cap A_m$ is semi-closed in X and as $\{B \cap A_m : m \in M\}$ is strongly nbd-finite, $\bigcup_m (B \cap A_m)$ is semi-closed in X which implies B is semi-closed in X.

If $B \cap A_m$ is semi-open in A_m, it can be proved , by considering complements, that B is semi-open in X.

This completes the proof.

4.4 : ALMOST CONTINUOUS, δ- CONTINUOUS AND SET CONNECTED MAPPINGS

Definition : 4.4

A mapping $f : X \to Y$ is said to be almost continuous if it satisfies one of the following conditions :

(a) If for each $x \in X$ and each open set V containing f(x), the closure of $f^{-1}(V)$ is a neighbourhood of X. (Husain [10]).

(b) If for each $x \in X$ and each open set $V \subset Y$ containing f(x), there exists an open set $U \subset X$ containing x such that $f(U) \subset Int(Cl(V))$. (Singal and Singal [31]).

From the above definition, we established the following proposition :

Proposition : 4.4.1

For a pre-open mapping $f : X \to Y$, Definition-4.4.1(b) implies definition-4.4.1(a).

Proof : Let $x \in X$ and let $V \subset Y$ be an open set containing f(x). then $f^{-1}(Cl(V)) \subset Cl(f^{-1}(V))$. (Levine [12]). Now we observe that Int(Cl(V)) is a regular open set and that $V \subset Int(Cl(V)) \subset Cl(V)$. Since Definition-4.4.1(b) is hold by f, $f^{-1}(Int(Cl(V))$ is open in X. Thus $f^{-1}(Int(Cl(V))) \subset f^{-1}(Cl(V)) \subset Cl(f^{-1}(V))$ and consequently $Cl(f^{-1}(V))$ is a neighbourhood of x. Thus f holds Definition-4.4.1(a).

Proposition : 4.4.2

Let $f : X \dashrightarrow Y$ be an almost continuous (Definition-4.4.1(b)) and let $V \subset Y$ be pre-open. If $x \notin f^{-1}(V)$, but $x \in Cl(f^{-1}(V))$, then $f(x) \in Cl(V)$.

<u>Proof</u> : Let $x \in X$ such that $x \notin f^{-1}(V)$ but $x \in Cl(f^{-1}(V))$ and suppose $f(x) \notin Cl(V)$. Then there exists an open set W such that $f(x) \in W$ and $W \cap V = \phi$. Since V is pre-open, then $Cl(W) \cap V = \phi$ and $Int(Cl(W)) \cap V = \phi$. Since f holds definition-4.4.1(b), there exists an open set $U \subset X$ such that $x \in U$ and $f(U) \subset Int(Cl(W))$. Consequently, $f(U) \cap V = \phi$. However, since $x \in Cl(f^{-1}(V))$, $U \cap f^{-1}(V) \neq \phi$ so that $f(U) \cap V \neq \phi$. So, we have a contradiction. It follows that $f(x) \in Cl(V)$.

Proposition : 4.4.3

Let $f : X \dashrightarrow Y$ be an almost continuous mapping (Definitin-4.4.1(b)). Then for each pre-open $V \subset Y$, $Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$.

Proof : Let $V \subset Y$ ba a pre-open set. By Proposition-4.4.2, $f(Cl(f^{-1}(V))) \subset Cl(V)$. Since,

$Cl(f^{-1}(V)) \subset f^{-1}(f(Cl(f^{-1}(V))))$. For any mapping.

we have, $Cl(f^{-1}(V)) \subset f^{-1}(f(Cl(f^{-1}(V)))) \subset f^{-1}(Cl(V))$.

Definition : 4.4.2

A mapping $f : X \to Y$ is said to be weakly continuous if for each point $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subset Cl(V)$. (Levine [12]).

Proposition : 4.4.4

Let $f : X \to Y$ is weakly continuous and pre-open mapping. Then f is almost continuous.

Proof : Let $x \in X$ and V be an open set of Y containing $f(x)$. Since f is weakly continuous, there exists an open set U of X containing x such that $f(U) \subset Cl(V)$. Since f is pre-open and U is open in X, then $f(U)$ is pre-open and hence,

$f(U) \subset Int(Cl(f(U))) \subset Int(Cl(V))$.

Hence f is almost continuous by Definition-4.4.1(b).

Definition : 4.4.3

A mapping $f : X \to Y$ is said to be δ-continuous iff for each $x \in X$ and each regular open set V containing $f(x)$, there exists a regular open set U containing x such that $f(U) \subset V$. (Noiri [26]).

Proposition : 4.4.5

If a mapping $f : X \to Y$ is δ-continuous and X_0 is pre open in X. Then $f/X_0 : X_0 \to Y$ is δ-continuous.

<u>Proof</u> : Let $x \in X_0$ and $V \in RO(Y)$ containing $f(x)$. Since f is δ- continuous, there exists $U \in RO(X)$ containing x such that $f(U) \subset V$ and hence $U \cap X_0 \in RO(X_0)$. Hence there exists $H = U \cap X_0 \in RO(X_0)$ containing x such that $(f/X_0) \subset V$. Therefore, f/X_0 is δ- continuous.

Definition : 4.4.4

A mapping $f : X \to Y$ is called weakly open if $f(U) \subset Int(f(Cl(U)))$ for each open set $U \subset X$. (Rose [30]).

Proposition : 4.4.6

If a mapping $f : X \to Y$ is weakly open, δ- continuous , then f is δ- continuous.

Proof : Let $x \in X$ and $V \subset Y$ be an open set containing $f(x)$. Then there exists an open set $U \subset X$ containing x such that $f(Cl(U)) \subset Cl(V)$. Since f is weakly open, $f(Int(Cl(U))) \subset Int(f(Cl(Int(Cl(U))))) \subset Int(f(Cl(U)))$. Then $f(Int(Cl(U))) \subset Int(Cl(V))$ and hence f is δ-continuous.

Proposition : 4.4.7

If $f : X \to Y$ is weakly continuous, then $f^{-1}(V) \subset Int(f^{-1}(Cl(V)))$ for each pre-open set V of Y.

Proof : Let V be a pre-open set of Y, then $V \subset Int(Cl(V))$. Since f is weakly continuous, then for each point $x \in f^{-1}(V) \subset f^{-1}(Int(Cl(V)))$, there exists an open set W_x such that $x \in W_x$ and $f(W_x) \subset Cl(Int(Cl(V))) \subset Cl(V)$. Then $W_x \subset f^{-1}(Cl(V))$. Consequently

$f^{-1}(V) = \cup \{ x : x \in f^{-1}(V) \} = \cup \{ W_x : x \in f^{-1}(V) \}$ $\subset f^{-1}(Cl(V))$. Hence $f^{-1}(V) \subset Int(f^{-1}(Cl(V)))$.

Proposition : 4.4.8

If a mapping $f : X \to Y$ is weakly continuous, then $Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for each pre-open set V of Y.

<u>Proof</u> : Let $x \in Cl(f^{-1}(V))$ and let $x \notin f^{-1}(Cl(V))$. then there exists an open set W containing $f(x)$ such that $W \cap V = \phi$. Hence $Cl(W \cap V) = \phi$. Since V is pre-open. then

$$V \cap Cl(W) \subset Int(Cl(V)) \cap Cl(W) \subset Cl(Int(Cl(V)) \cap W)$$
$$= Cl(Int(Cl(V) \cap W)) \subset Cl(Int(Cl(V \cap W)))$$
$$\subset Cl(V \cap W) = \phi .$$

Since f is weakly continuous and W is an open set containing $f(x)$, there exists an open set U containing x such that $f(U) \subset Cl(W)$. Then $f(U) \cap V = \phi$. Also $x \in Cl(f^{-1}(V))$ and U is an open set containing x, then $U \cap f^{-1}(V) \neq \phi$, this implies $f(U) \cap V \neq \phi$, which gives us a contradiction. Therefore,

$$Cl(f^{-1}(V)) \subset f^{-1}(Cl(V)).$$

Definition : 4.4.5

A mapping $f : X \to Y$ is said to be set connected due to Kwak [11], iff for any closed open set F of $f(x)$, $f^{-1}(F)$ is closed-open in X.

Proposition : 4.4.9

Let Y be an extremally disconnected space. If a mapping $f : X \to Y$ is set connected, then f is almost continuous.

proof : Let x be any point of X and V be any set of Y containing $f(x)$. Since Y is extremally disconnected, $Cl(V)$ is closed-open in Y. Thus $Cl(V) \cap f(X)$ is closed-open in the subspace $f(X)$. Putting $f^{-1}(Cl(V) \cap f(X)) = U$. Since f is set connected, U is closed-open in X (Noiri [22], Lemma-1). Therefore, U is an open neighbourhood of x in X such that,

$$f(U) \subset Cl(V) \subset Int(Cl(V))$$

Hence f is almost continuous.

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