Total No. of Printed Pages—7

**3 SEM TDC MTMH (CBCS) C 5**

**2022**

( Nov/Dec )

MATHEMATICS

( Core )

Paper : C–5

**( Theory of Real Functions )**

*Full Marks* : 80

*Pass Marks* : 32

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

**1.** *(a)* State the divergence criteria of a limit of a function. 1+1=2

*(b)* Define cluster point of a set with an example. 1+1=2

*(c)* Use ε-δ definition to establish that

$$\lim_{x \to c} x^2 = c^2$$ 2

*(d)* Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, a cluster point of $A$. Show that if $f$ has a limit, when $x \to c$, then $f$ is bounded. 3

*(e)* Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, a cluster point of $A$. If $a \le f(x) \le b$, $\forall\ x \in A$ and $x \ne c$, and $\lim_{x \to c} f(x)$ exists, then show that

$$a \le \lim_{x \to c} f(x) \le b$$ 3

*(f)* State and prove squeeze theorem. 1+3=4

*(g)* Show by using definition that

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$ 3

*(h)* Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and $c \in A$. Then establish any *one* of the following : 3

*(i)* If $f$ is continuous at $c \in A$, then given any $\varepsilon$-neighbourhood $V_\varepsilon(f(c))$ of $f(c)$, $\exists$ a $\delta$-neighbourhood $V_\delta(c)$ of $c$, such that if $x \in A \cap V_\delta(c)$, then

$$f(x) \in V_\varepsilon(f(c)).$$





*(ii)* Let given any $\varepsilon$-neighbourhood $V_\varepsilon(f(c))$ of $f(c)$, $\exists$ a $\delta$-neighbourhood $V_\delta(c)$ of $c$, such that if $x \in A \cap V_\delta(c)$, then $f(x) \in V_\varepsilon(f(c))$. Then $f$ is continuous at $c \in A$.

*(i)* Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and define $|f|$ by $(|f|)(x) = |f(x)|$, $\forall\ x \in A$. Show that if $f$ is continuous at $c \in A$, then $|f|$ is also continuous at $c \in A$. 4

*Or*

Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and $f(x) \ge 0$, $\forall\ x \in A$. Defined $\sqrt{f}$ by $(\sqrt{f})(x) = \sqrt{f(x)}$, $\forall\ x \in A$. Show that if $f$ is continuous at $c \in A$, then $\sqrt{f}$ is continuous at $c$.

*(j)* State and prove location roots theorem. 1+3=4

*Or*

Let $I$ be a closed and bounded interval, and $f : I \to \mathbb{R}$ is continuous on $I$. Then show that $f : I \to \mathbb{R}$ is uniformly continuous. 4

**2.** *(a)* Define relative maximum of a real-valued function at a point. 1

*(b)* State the first derivative test for the relative maximum at a point of a real-valued function. 1

*(c)* Show that if $f : I \to \mathbb{R}$ is differentiable and $f(x) \geq 0$, $\forall\ x \in I$, then $f$ is increasing on $I$. 2

*(d)* Using first derivative test, show that $f(x) = x^2$ has a minima at $x = 0$. 2

*(e)* State and prove the interior extremum theorem. 3

*Or*

Let $f : I \to \mathbb{R}$ be differentiable at $c$. If $f'(c) < 0$, then show that

$$f(x) > f(c), \quad \forall\ x \in (c - \delta,\ c)$$

*(f)* State and prove Caratheodory's theorem. 4





*(g)* Use mean value theorem to show that if $f(x) = \sin x$ which is differentiable, $\forall\ x \in \mathbb{R}$, then

$$|\sin x - \sin y| \leq |x - y| \quad \forall\ x,\ y \in \mathbb{R}$$ 4

*Or*

Use mean value theorem to show that

$$-x \leq \sin x \leq x \quad \forall\ x \geq 0$$

*(h)* State and prove the mean value theorem. 4

*(i)* State and prove Darboux's theorem. 4

*Or*

Use mean value theorem to show that

$$e^x \geq 1 + x \ \forall\ x \in \mathbb{R}$$

and hence show that $e^\pi > \pi^e$.

**3.** *(a)* Define a convex function on an interval and give its geometrical interpretation. 1+1=2

(b) Show that the function $f(x) = x^3$ has no relative extremum at $x = 0$. 2

(c) Show that

$$f(x) = x + \frac{1}{x};\ x > 0$$

is a convex function. 3

(d) Determine relative extrema of the function

$$f(x) = x^4 + 2x^3 - k$$

where $k$ is a constant. 3

(e) State and prove Cauchy's mean value theorem. 5

(f) State and prove Taylor's theorem with Lagrange's form of remainder. 5

(g) Define Taylor's and Maclaurin's series. Obtain Maclaurin's series for the function $\sin x$. 2+3=5





*Or*

Show that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{\lfloor 2n}\ \forall\ x \in \mathbb{R}$$ 5

★★★