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4 SEM TDC MTMH (CBCS) C 9

2023

(May/June)

MATHEMATICS

(Core)

Paper : C–9

(Riemann Integration and Series of Functions)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* State the two vital requirements for existence of

$$\int_a^b f(x)\,dx$$ 1+1=2

(b) Show that if $f \in R[a, b]$, then the value of $\int_a^b f(x)dx$ is unique. 3

Or

Show that every constant function is integrable.

2. (a) Let $P = \{([x_{i-1}, x_i]), t_i\}_{i=1}^n$ be a tagged partition of $I = [a, b]$. Then define Riemann sum of $f : [a, b] \to \mathbb{R}$. Give an example of the Riemann sum if $I = [1, 2]$. 2

(b) Let $P = \{([x_{i-1}, x_i]), t_i\}_{i=1}^n$ be a tagged partition of $I = [a, b]$. Then show that $S(kf, P) = kS(f, P)$. 3

(c) Answer any *four* questions from the following : 5×4=20

(i) Write an example with explanation thereof that all bounded functions are not Riemann integrable.

(ii) Let $f : [a, b] \to \mathbb{R}$ is such that if $x_1 < x_2$, then $f(x_1) \le f(x_2)$. Show that $f \in R[a, b]$.

(iii) Let $f : [a, b] \to \mathbb{R}$ be integrable. Then $|f|$ is integrable and show that

$$\left| \int_a^b f(x)dx \right| \le \int_a^b | f(x) | dx$$

(iv) Let $f, g : [a, b] \to \mathbb{R}$ be integrable and $f(x) \le g(x) \ \forall \ x \in [a, b]$. Then show that

$$\int_a^b f(x)dx \le \int_a^b g(x)dx$$

(v) Let $f : [a, b] \to \mathbb{R}$ be integrable. Define F on $[a, b]$ as $F(x) = \int_a^x f(t)dt$ where $x \in [a, b]$. Show that F is differentiable at $c \in [a, b]$ and $F'(c) = f(c)$.

3. (a) Show that

(i) $\Gamma(1)=1$

(ii) $\Gamma(n+1)=n\Gamma(n)$ 1+2=3

(b) Show that if $m \in \mathbb{N}$, then $\Gamma(m+1)=\underline{|m}$. 3

(c) Discuss the convergence of beta function. 4

Or

Show that $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$ and hence show that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

4. (a) State whether true or false : 1

Pointwise convergence implies uniform convergence.

(b) Let (f_n) be a real sequence of functions defined on a finite set $X=\{a_1, \dots, a_k\}$ converging pointwise to a function $f: X \to \mathbb{R}$. Establish that the convergence is uniform. 2

(c) Let (f_n) be a sequence of integrable functions on $[a, b]$. Let $f_n \to f$ uniformly on $[a, b]$. Show that f is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \lim \int_a^b f_n(x)\,dx$$ 4

(d) Show that if (f_n) be a uniformly Cauchy sequence on a set X in $\mathbb{R}$, then it converges to $f: X \to \mathbb{R}$ uniformly. 4

(e) Show that the series

$$\sum_{n=1}^{\infty} \frac{x}{(1+nx^2)n}$$

converges uniformly on any interval $[a, b]$. 4

(f) State and prove Cauchy's criterion for the uniform convergence of a series. 5

(g) Let $f_n : (a, b) \to \mathbb{R}$ be differentiable and the sequence (f_n') converges uniformly to $g : (a, b) \to \mathbb{R}$. Let there exists $c \in (a, b)$ such that the sequence $(f_n(c))$ converges. Then show that the sequence (f_n) converges uniformly to a continuous function $f : (a, b) \to \mathbb{R}$. 5

5. *(a)* State whether true or false : 1

A power series is a particular case of infinite series of functions

$$\sum_{n=0}^{\infty} f_n(x)$$

(b) Let $\sum_{n=0}^{\infty} a_n (x-a)^n$ be a power series. Show that there exists a unique extended real number R; $0 \le R < \infty$, such that $\forall x$ with $|x-a| < R$, the series converges absolutely and uniformly to a function f on $(-r, r)$; $0 < r < R$. 4

(c) Given a power series $\sum_{n=0}^{\infty} a_n (x-a)^n$, determine an extended real number R such that $\frac{1}{R} = \lim |a_n|^{\frac{1}{n}}$. 5

(d) State and prove Abel's limit theorem. 5

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