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4 SEM TDC MTMH (CBCS) C 10

2023

(May/June)

MATHEMATICS

(Core)

Paper : C–10

(Ring Theory and Linear Algebra—I)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Define unit element in a ring. Give an example. 1+1=2

(b) Let R be a ring and $a, b \in R$. Show that $a(-b) = (-a)b$. 2

(c) Show that intersection of two subrings of a ring R is a subring of R. 3

(d) Show that the ring $\mathbb{Z}_p$ of integers modulo p (p being a prime) is a field. 3

(e) Let R be a commutative ring with unity and let A be an ideal of R. Show that $\frac{R}{A}$ is a field if and only if A is maximal. 5

Or

Define a principal ideal domain. Show that the ring $\mathbb{Z}$ of integers is a principal ideal domain. 1+4=5

(f) Define characteristic of a ring. Let R be a ring with unity. Then show that

characteristic of $R =$

$$\begin{cases} n, \text{ if 1 has order } n \text{ under addition} \\ 0, \text{ if 1 is of infinite order under addition} \end{cases}$$

1+2+2=5

Or

Prove that the ring

$$\mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}, d \text{ is a positive integer}\}$$

is a field. 5

2. (a) Give an example of ring homomorphism. 1

(b) Is the map $f : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ such that $f(x) = 2x$ a ring homomorphism? Justify. 2

(c) Let $f : R \to S$ be a homomorphism from ring R to ring S. If $\ker f = \{0\}$, then show that f is one-one. 2

(d) Let $f : R \to S$ be a ring homomorphism from ring R to ring S. Show that kernel of f is an ideal of R. 2

(e) Determine all ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$, $\mathbb{Z}$ being the ring of integers. 3

Or

Show that if R is a ring with unity and char $(R) = n > 0$, then R contains a subring isomorphic to $\mathbb{Z}_n$.

(f) Show that if $f : R \to S$ is an onto homomorphism, then

$$S \cong \frac{R}{\ker f}$$ 5

Or

State the second theorem of ring isomorphism. Show that

$$\frac{\mathbb{Z}}{<2>} \cong \frac{5\mathbb{Z}}{10\mathbb{Z}}$$ 1+4=5

3. *(a)* Define a vector subspace. 1

(b) Is $Q(\mathbb{R})$ a vector space? 1

(c) Is the set $\{(1, 2, 5), (2, -1, 0), (7, -1, 5)\}$ linearly independent over $\mathbb{R}$? Justify. 2

(d) Let $S = \{e_1 - e_2, e_1 + e_2\} \in \mathbb{R}^2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Find $L(S)$. 2

(e) Show that any basis of a vector space is a linearly independent set. 2

(f) Show the sum of two subspaces of a vector space V is a subspace of V. 3

(g) Let V be a vector space and W_1 and W_2 be two subspaces of V. Show that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$ 4

Or

Let V be a finite dimensional vector space and W be a subspace of V. Show that $\dim V / W = \dim V - \dim W$.

4. *(a)* Define a linear map. Show that the map $T : \mathbb{R}^3 \to \mathbb{R}$ such that

$$T(x, y, z) = x^2 + y^2 + z^2$$

is not linear. 1+2=3

(b) Show that if $T : V \to W$ is a linear map from vector space V to vector space W, then—

(i) $T(0) = 0$

(ii) $T(V)$ is a subspace of W 2+2=4

(c) Extend the vector (1, 1, 1) in $\mathbb{R}^3(\mathbb{R})$ to form a basis of $\mathbb{R}^3$. 3

(d) Answer any *four* of the following : 5×4=20

(i) Let V and W be two vector spaces and $T : V \to W$ be a linear map. Then show that

$$\dim V = \text{rank } T + \text{nullity } T$$

(ii) Let $\{v_1, \cdots, v_n\}$ be a basis of the vector space V and $w_1, w_2, \cdots, w_n$ be vectors in the vector space W. Show that there exists a unique linear map $T : V \to W$ such that

$$T(v_i) = w_i,\ i = 1, 2, \cdots, n$$

(iii) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T(x, y) = (x, x+y, y)$$

Then find the range of T.

(iv) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator defined by

$$T(x, y, z) = (3x, x-y, 2x+y+z)$$

Show that T is invertible. Also find the inverse map T^{-1}.

(v) Let the linear map $T : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by

$$T(x, y) = (x, y, x+y, x-y)$$

Find the matrix of T with respect to the standard bases.

(vi) Let $T : V \to W$ be an invertible linear map from vector space V to vector space W where

$$\dim V = \dim W = n$$

Show that T is non-singular if and only if T is onto.

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