Total No. of Printed Pages—7

3 SEM TDC MTMH (CBCS) C 5

2023

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C–5

(Theory of Real Functions)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Give an example of a proper subset of $\mathbb{R}$, whose cluster points are the elements of the proper subset itself. 1

(b) State whether true or false : 1

In the definition of $\lim\limits_{x \to c} f$ where c is a cluster point of the domain of f, it is immaterial whether f is defined at c or not.

(c) Use the definition of limit of a function to show that

$$\lim_{x \to c} x = c$$ 1

(d) Use ε-δ definition to establish that

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0$$ 2

(e) Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and c is a cluster point of A. Show that if $\lim_{x \to c} f(x) = L$, then $\lim_{x \to c} |f(x) - L| = 0$. 2

(f) Define a bounded function with a suitable example. 2

(g) Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and c is a cluster point of A. If $\lim_{x \to c} f < 0$, then show that there exists a neighbourhood $V_\delta(c)$ of c such that $\forall x \in A \cap V_\delta(c)$ with $x \neq c$, $f(x) < 0$. 3

Or

Use squeeze theorem to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

(h) Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions where $A, B \subseteq \mathbb{R}$ and $f(A) \subseteq B$. If f is continuous at $c \in A$ and g is continuous at $b = f(c) \in B$, then show that the composition $gf : A \to \mathbb{R}$ is continuous at $c \in A$. 3

(i) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b] \subseteq \mathbb{R}$. Then show that f is bounded on $[a, b]$. 3

(j) Let $f : I \to \mathbb{R}$ be continuous on I, an interval. If $a, b \in I$ and $k \in \mathbb{R}$ satisfy $f(a) < k < f(b)$, then show that there exists a point $c \in I$ between a and b such that $f(c) = k$. 4

(k) Let $f : I \to \mathbb{R}$ be continuous where I is a closed bounded interval in $\mathbb{R}$. Show that the set $f(I) = \{f(x) : x \in I\}$ is a closed bounded interval. 4

(l) Use sequential criteria of continuity to establish that Dirichlet's function is not continuous at any real numbers. 4

2. *(a)* State Caratheodory's theorem. 1

(b) State whether true or false : 1

Let x_0 be an interior point of an interval I and the derivatives $f', f'', \dots, f^{(n)}$ exist and continuous in a neighbourhood of x_0 with $f'(x_0) = f''(x_0) = \dots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$. If n is odd, then f has no relative extremum at x_0.

(c) Use first derivative test to establish that f defined by $f(x) = x^3$ has no extremum at $x = 0$. 2

(d) Find the relative extremum of the function $f(x) = \sum_{i=1}^{n} (a_i - x)^2$ where $a_i \in \mathbb{R}$; $1 \leq i \leq n$. 2

(e) Let f be continuous on an interval $[a, b]$ and differentiable on (a, c) and (c, b) where $c \in (a, b)$. Then if there exists a neighbourhood $(c - \delta, c + \delta)$ of c in $[a, b]$ such that $f'(x) \geq 0 \ \forall x \in (c - \delta, c)$ and $f'(x) \leq 0 \ \forall x \in (c, c + \delta)$, then show that f has a relative maximum at c. 3

Or

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. Let $c \in I$ and $f'(c)$ exist and positive. Then show that there exists a number $\delta > 0$ such that $f(x) > f(c)$ $\forall x \in (c, c + \delta)$.

(f) Let f be continuous on a closed interval $[a, b]$ and differentiable on an open interval (a, b). If $f'(x) = 0 \ \forall x \in (a, b)$, show that f is constant on $[a, b]$. Hence, show that if g is another function satisfying that it is continuous on $[a, b]$ and differentiable on (a, b) with $f'(x) = g'(x) \ \forall x \in (a, b)$, then there exists a constant k such that $f(x) = g(x) + k$ on $[a, b]$. 4

(g) Use mean value theorem to show that $\frac{x-1}{x} < \log x < x - 1$ for $x > 1$. 4

(h) State and prove Rolle's theorem and give its geometrical interpretation. 4

(i) Use Taylor's theorem to show that

$$1 - \frac{x^2}{2} \leq \cos x \ \forall \ x \in \mathbb{R}$$ 4

Or

Show that if $\alpha > 1$, then

$$(1+x)^{\alpha} > 1 + \alpha x \;\forall\; x > -1$$

and $x \neq 0$.

3. (a) Consider Cauchy's mean value theorem for two functions f and g which are continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) \neq 0 \;\forall\; x \in (a, b)$. For what value of $g(x)$, Cauchy's mean value theorem reduces to mean value theorem? 1

(b) State Lagrange's form of remainder in Taylor's theorem for a function f defined on $[a, b]$. 1

(c) State Maclaurin's infinite series expansion about $x = 0$ mentioning the interval of expansion. 2

(d) Investigate whether the function $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = x \log x$ is convex or not. 2

(e) Investigate the function

$$f(x) = (x-3)^5 (x+1)^4$$

for relative extrema. 4

(f) Let $f : I \to \mathbb{R}$ have second derivative on an open interval I of $\mathbb{R}$. Show that f is a convex function on I if and only if $f''(x) \geq 0 \;\forall\; x \in I$. 5

(*Continued*)

(g) Expand $\cos x$ in the Maclaurin's series. 5

Or

Expand $\log(1+x)$ in the Maclaurin's series.

(h) State and prove Taylor's theorem with Cauchy's remainder. 5

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